CHAPTER 7A&B

7-1. The White Dwarf Stars:

The first indication that white dwarf stars existed came with the visual verification of a companion of the star Sirius in 1862, though Bessel had discovered there was a companion years earlier from an analysis of the proper motion of Sirius. From an analysis of the orbital dynamics of the pair, it was found that Sirius B has a mass equal to $1.05M_\odot$ and the spectrum indicated that its surface temperature was about 28,000K. However, the observed luminosity of the star was $0.03L_\odot$. From the Stefan-Boltzmann Law one derives a radius of 5.5×10^8 cm which is about $0.008R_\odot$. This meant that 1.05 solar masses were compacted into an object the size of the Earth. In turn, this leads to a mean density for the star of 3.0×10^6 g/cm^3, which, at that time, was viewed as quite extraordinary. We now estimate what the pressure is at the center of the star using the hydrostatic equilibrium equation .

$$dP/dr = -GM\rho/r^2 = -G(4/3\pi r^3\rho)\rho/r^2 = -4/3\pi G\rho^2 r \qquad (6\text{-}30)$$

We integrate this from the surface where P=0 and r=R to the center of the star where r=0, assuming the density is constant.

$$\Delta P = \int dP = -\frac{4}{3}\pi G\rho^2 \int_{r=R}^{r=0} r\,dr = -\frac{4}{3}\pi G\rho^2 \frac{r^2}{2}\Big|_R^0$$

$$\Delta P = P_c - P_{surf} = -2/3\pi G\rho^2[0-R^2] = 2/3\pi G\rho^2 R^2. \qquad (6\text{-}31)$$

Take the surface Pressure to be 0 and we have

$$P_c = 2/3\pi G\rho^2 R^2 \approx 3.5 \times 10^{23} \text{ dynes/cm}^2.$$

This is a fantastic pressure. We can now estimate the central temperature from the equation of state:

$$T_c = P_c \mu m_H/\rho k = 7 \times 10^8 \text{ K}. \qquad (5\text{-}69)$$

More exact models with better density functions yield core temperatures of about 7×10^7 K. At this temperature, any TNF reactions would produce a luminosity far higher than that observed. Hence, we conclude there are no nuclei in the core of the star that may undergo fusion. We now realize that such stars consist mainly of carbon and oxygen nuclei with shallow upper layers of H and He.

Characteristics of white dwarfs:

1. Mass is less than the Chandrasekhar Limit of about $1.4M_\odot$, but this depends on the molecular weight.

2. Degenerate electron pressure balances gravity.

3. The star has its minimum radius and can not contract any farther. Sizes range 6000 Km ($0.01R_\odot$ or 1 Earth radius) to 150,000 Km.

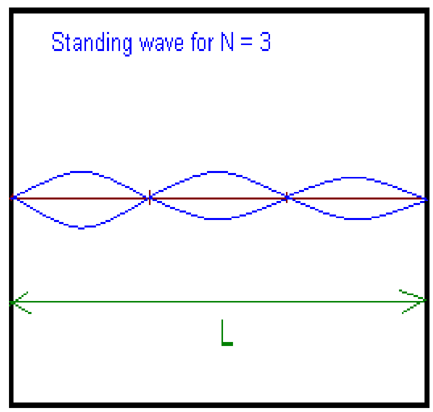
4. Typical density is 10^6 g/cm^3.

5. Surface temperatures range from 50,000K to 5,000K.

6. There is no internal source of energy and the star is slowly cooling along a line of constant radius in the H-R Diagram to become a black dwarf, but this will take about 50 billion years. It will take 10 billion years just to cool to 3000K.

7. The atomic nuclei are constrained in place forming a crystal lattice often referred to as a quantum solid.

In a completely ionized gas, the electrons exert a separate pressure from that of the nuclei. The electrons are fermions and obey the Pauli Exclusion Principle. As the temperature of a gas decreases, fermions will occupy the lowest energy levels first and then successively occupy the higher energy states that are available. As T approaches zero, the motions of the electrons in the excited states produce the electron pressure. At T=0, all of the lower energy states and none of the higher states are occupied. Such a fermion gas is said to be completely degenerate and the energy dividing the occupied states from the vacant states is called the Fermi Energy, ε_F. The latter may be found from quantum statistical theory as follows (or see Sears and Salinger, pp. 407-410):



The box is 3 dimensional

Electrons or fermions may be represented as de Broglie standing waves in a volume of dimension L. L is really the mean separation of the fermions in the gas. The volume is actually the potential well in which the fermion finds itself as a result of the repulsive coulombic forces of the other fermions. We choose rectangular coordinates to simplify matters. Then take the volume to be a cube. The wavelength of the electrons is $\lambda_i = 2L/N_i$ for each of the three sides of the cube. The N_i are integer quantum numbers associated with each dimension, i = x, y, or z. So when $N_i = 1$, we have the maximum standing wave length of 2L along either the x, y, or z directions. The de Broglie wavelength is related to the momentum of the electron by the relation

$$p_i = hN_i/2L \tag{6-32}$$

and the total kinetic energy of the particle is $\varepsilon = p^2/2m_e$, where p is the total momentum

$$p^2 = \sum_{i=1}^{3} p_i^2.$$

Hence, the kinetic energy of a fermion is:

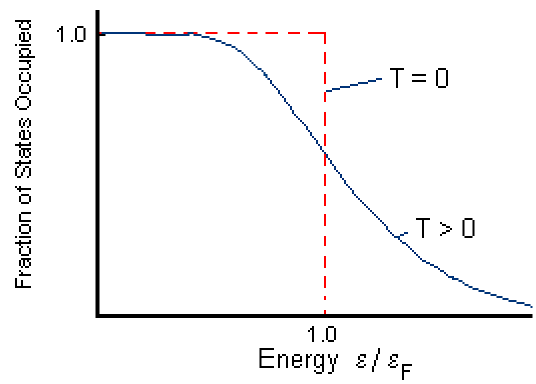
$$\varepsilon_F = \left(\frac{h^2}{8m_eL^2}\right)\left(\sum_i N_i^2\right) = \frac{h^2N^2}{8m_eL^2} \tag{6-33}$$

The total number of electrons in an electron gas is N_e is equal to the total number of sets of unique quantum numbers, N_x, N_y, N_z times 2. The factor of 2 comes about because 2 electrons can have the same set of quantum numbers if they have opposite spins, which means there are really four quantum numbers, one for the spin. Now N may be thought of as the radius of quantum number space and so the volume of this space is $(4/3)\pi N^3$, and N_x, N_x, and N_z are the coordinates of each electron in this space. But the quantum numbers cannot be negative, so the electrons only occupy an octant of this quantum space which has volume (1/8) (4/3)πN^3. So the number of electrons that can occupy this space is twice this because of the spins. Hence,

$$N_e = 2(\tfrac{1}{8})(4/3)\pi N^3$$

Now solve this equation for N and we get $N = \left(\frac{3N_e}{\pi}\right)^{1/3}$. Substitute this into (6-33) and simplifying with $\hbar = h/2\pi$ and $n_e = N_e/L_3$ we get:

$$\varepsilon_F = \frac{\hbar^2}{2m_e}\left(3\pi^2 n_e\right)^{2/3} \tag{6-34}$$



The Fermi energy is the maximum energy that any electron can have at T = 0. As the temperature rises above 0, some fermions will occupy states with energies greater than the Fermi energy. The gas is then partly degenerate. This is illustrated in the accompanying diagram. However, when n is very large, such as in a white dwarf, degeneracy is a good approximation. That is, all but the most energetic fermions will have energies less than the Fermi energy.

So, in white dwarfs, electrons are as densely packed as permitted by the Pauli Exclusion Principle. Under this condition, the electrons act as if they are in one enormous atom. Since the electrons occupy their lowest energy levels, they can not undergo transitions to produce photons or radiate electromagnetic waves. Hence, the white dwarf is effectively very cold in spite of the high internal energy it holds.

Now we want to express the Fermi energy in terms of the temperature and density. Start with

$$n_e = (\#\ \text{electrons/nucleon})(\#\ \text{nucleons/volume}) = (Z/A)(\rho/m_H).$$

So: $$\varepsilon_F = (\hbar^2/2m_e)[3\pi^2(Z/A)\,\rho/m_H]^{2/3}. \tag{6-35}$$

This is the maximum energy of electrons at T=0. Now the thermal energy of the electrons is (3/2)kT. If this is < ε_F, the electrons cannot undergo a transition to an unoccupied state and the gas is degenerate. Hence, we have the condition for degeneracy that (3/2)kT < ε_F or

$$T/\rho^{2/3} < (\hbar^2/3km_e)[3\pi^2 Z/Am_H]^{2/3} \tag{6-36}$$

On the average, Z= (1/2)A, so Z/A= 1/2. Then the above becomes:

$$T/\rho^{2/3} < (\hbar^2/3km_e)[3\pi^2/2m_H]^{2/3} = 1.3 \times 10^5\ \text{K cm}^2\ \text{g}^{2/3}$$

Let us refer to this quantity as DC, the degeneracy criterion. That is

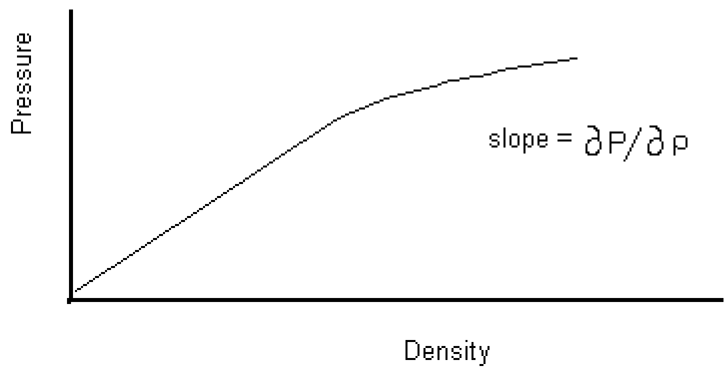
$$T/\rho^{2/3} < DC = 1.3 \times 10^5\ \text{K cm}^2\ \text{g}^{2/3} \tag{6-37}$$

The smaller the value of $T/\rho^{2/3}$, the greater the degeneracy of the gas.

Let us calculate the value of $T/\rho^{2/3}$ for the core of the Sun, where T is 15 x 10^6 K and ρ_c = 162 g/cm^3. This gives a value of 5.3 x 10^5, which is > DC. So degeneracy plays a small role in the core of the Sun. Degenerate electron pressure contributes only a few tenths of a percent of the central pressure in the Sun. As the Sun evolves, the core contracts and the density increases faster than T, because of radiative losses and the degeneracy will become more significant.

If we use the values of T and ρ in the core of Sirius B, we get a value smaller than DC. Hence, in the interior of Sirius B, degenerate electron pressure dominates and supports the star from further gravitational contraction.

It was S. Chandrasekhar, between 1931 and 1932, who worked out the theory of white dwarfs. He found that, as a consequence of special relativity, the rate of change of pressure with density would decrease at very high densities. See the diagram below, where the slope of the curve is decreasing with increasing density. This meant that a maximum stable mass existed for stars that



have exhausted their internal energy sources. In 1932, Landau showed that more matter than the critical value would lead to a collapse of the star without limit.

When a star exhausts most of its nuclear fuel, the star is expected to be composed either of nuclei embedded in a gas of electrons (a white dwarf with densities in the range of 10^5 to 10^8 g/cm^3) or a gas of neutrons in equilibrium. The latter would be a neutron star with densities in the range of 10^{14} to 10^{16} g/cm^3. In both cases, the material is a degenerate Fermi gas held together by gravity.
The pressure required to balance gravity results from the zero-point energy, allowing for the Pauli Exclusion Principle rather than from the kinetic energy of an ideal gas. In this sense, the gas is acting like a solid though it is not a solid.

In a white dwarf, the main contribution to pressures comes from the gas of degenerate electrons whereas the nuclei contribute mainly to the density. In a neutron star, both pressure and density are generated by the gas of neutrons and there are no electrons.

Consider a cold star composed of particles of mass m and number density n. The particles comprising the gas are Fermions with spin ½, such as electrons, protons, neutrons, and mu-mesons. By the Pauli Exclusion Principle, each particle occupies a volume 1/n. From the Heisenberg uncertainty principle:

$$\Delta x \Delta p \approx h/2\pi. \tag{6-38}$$

Here p is momentum. So for each particle we have that $\Delta x^3 = 1/n_e$, and hence,

$$\Delta p_x \approx hn^{1/3}/2\pi \approx p_x. \tag{6-39}$$

If the gas is non-relativistic, $v_x << c$, then the velocity of a particle is

$$v_x \approx \Delta p_x/m = hn^{1/3}/2\pi m_e \qquad (6\text{-}40)$$

Now the pressure, as for a simple gas, is the time rate of change momentum per unit area, which may be expressed as:

$$P = \Delta pvn = hn^{1/3}/2\pi(hn^{1/3}/2\pi m_e)n = (h/2\pi)^2 n^{5/3}/m_e \qquad (6\text{-}41)$$

The mass here is appropriately the electron mass rather than the mass of the nuclei. This is because the electrons have much larger velocities and will therefore dominate the pressure. For a relativistic gas, $v_e \approx c$. Then

$$P \approx \hbar\, n^{1/3} cn = \hbar c\, n^{4/3} \qquad (6\text{-}42)$$

We shall now derive the mass limit for a white dwarf. Gravity is $F_g = GM\Delta m/R^2$. Now consider a unit volume of the star where $\Delta m = nm_n$, where m_n is the average mass of the nuclei. For hydrostatic equilibrium, gravity must be balanced by the pressure gradient (see Chapter 5)

$$P/R = GMnm_n/R^2, \qquad (6\text{-}43)$$

or

$$P = GMnm_n/R \qquad (6\text{-}44)$$

Now $M=\rho V = nm_nV = nm_nR^3$. Solving for R we have $R = [M/nm_n]^{1/3}$. Substitute this for R in the above and we get

$$P = GM^{2/3}(nm_n)^{4/3} \qquad (6\text{-}45)$$

In the non-relativistic regime, equilibrium is achieved when (6-45) equals (6-41), or

$$GM^{2/3}(nm_n)^{4/3} = \hbar^2\, n^{5/3}/m_e \qquad (6\text{-}46)$$

Now solve for M in (6-47):

$$M = G^{3/2}\hbar^3\, n^{1/2}\, m_e^{-3/2}\, m_n^{-2} \qquad (6\text{-}47)$$

In such a case, for any specified n, a value of M can always be found.

Relativistic Case:

For $v \cong c$, equate the left side of (6-46) with (6-42):

$$GM^{2/3}(nm_n)^{4/3} = \hbar c\, n^{4/3} \qquad (6\text{-}48)$$

Here n cancels. Solving for M we get:

$$M = (\hbar c/G)^{3/2}(m_n)^{-2} \qquad (6\text{-}49)$$

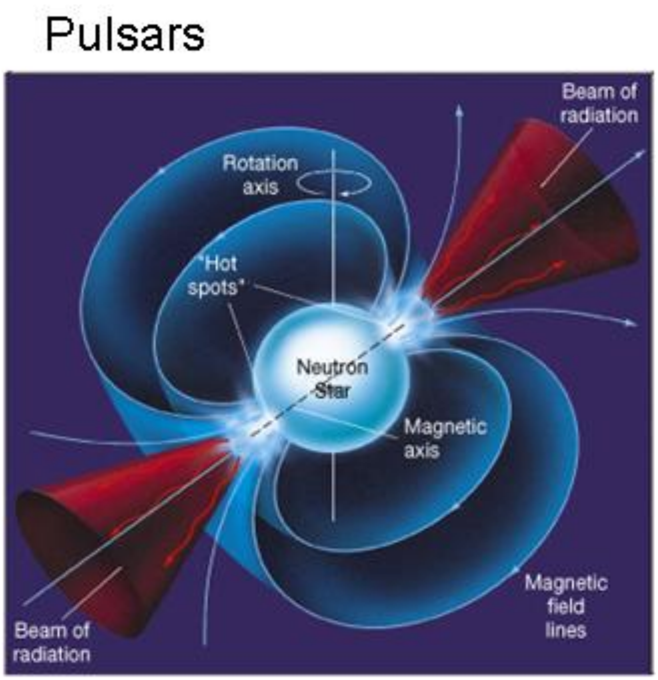
Or

$$M = 6.65M_{\odot}\mu^{-2} \cong 1.44M_{\odot} \qquad (6\text{-}50)$$

This is the Chandrasekhar limit. Note that M is independent of n. This means that the same mass is obtained independent of the radius. Stars more massive than this cannot be supported by electron degeneracy pressure, no matter how small the star.

7-2. Neutron Stars

Stars with masses between the Chandrasekhar limit and about $3M_{\odot}$ gravitationally collapse to the point where all the electrons are forced to combine with the protons to form neutrons. Eventually degenerate neutron pressure is able to halt further contraction. A $1.45M_{\odot}$ neutron star would consist of $1.45M_{\odot}/m_n \cong 10^{57}$ neutrons. It is in effect a huge atomic nucleus that is held together by gravity and supported by degenerate neutron pressure and has an atomic weight of 10^{57}. The typical radius of a neutron star is 10km with a density of 2.3×10^{14} g/cm^3. The acceleration of gravity at the surface of such an object would be 190 billion times greater than the acceleration of gravity at the Earth's surface. Upon collapsing, the conservation of angular momentum demands the star rotate with a period of a few milliseconds. Also, any imbedded magnetic field would be intensified upon collapse to about 10^8 Tesla. If the magnetic axis is tilted with respect to the rotational axis, the pulsar phenomenon is observed.



7-3. Black Holes

If the final core of a star exceeds $3M_{\odot}$, it will gravitationally collapse to the point where degenerate neutron pressure cannot balance gravity. In fact, there is no known force that can halt the collapse of the star to a singularity. As the core contracts, the surface gravity becomes so strong that that the escape velocity exceeds the velocity of light. Hence, no electromagnetic radiation can leave the object and it becomes black. From the general relativistic view, the object has curved the space-time continuum around itself to such a degree it is like a hole in space. Hence the term "black hole," which was coined by John Wheeler. The size of the star when it has contracted to the point where the escape velocity is equal to the velocity of light is called the "critical radius" or "Schwarzschild radius." This is given by

$$R_c = 2GM/c^2 \qquad (6\text{-}51)$$

Every body has a critical radius, even you. For the Sun, $R_c \cong 3$ km. Once a body has collapsed inside of its critical radius, the collapse is not reversible nor can it be halted. In addition, no energy can escape from it.

At $1.5R_g$, photons traveling tangential to the radius would be bent into an orbit around the star. They would be caught in a spherical cloud from which they slowly leak forever. Photons leaving at other angles can escape. At R_g, only photons traveling vertically can escape. They would have to do work against gravity and are therefore greatly red shifted, where $\Delta E = h\Delta\nu$.

General Relativity

In general relativity, the laws of physics are the geometry of the space-time continuum. The space-time continuum is elastic and is deformed by the presence of matter. Matter is thought to give the S-T continuum curvature. A line element in a 3 dimensional manifold may be written as:

$$ds^2 = dx^2+dy^2+dz^2.$$

In spherically symmetric space:

$$ds^2 = dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2.$$

As r approaches infinity, the space becomes flat.

We now introduce time:

$$ds^2 = A(r)dt^2 - B(r)\,dr^2 - C(r)r^2d\theta^2 - D(r)r^2\sin^2\theta d\varphi^2.$$

The functions A, B, C, and D define the curvature of the manifold or space. They are found by solving the Einstein field equations, which is very difficult to do. In 1916, two months after Einstein published his general theory of relativity, the German astronomer Karl Schwarzschild found a solution by assuming there is no angular momentum in the system and no magnetic field. The results are:

$$A = (1-2GM/c^2r) \qquad B = (1-2GM/c^2r)^{-1}$$
$$C = 1 \qquad D = 1$$

This is known as the **Schwarzschild metric**. M is the total mass in the system including mass-energy in the field. Only then is the inertial mass equal to the gravitational mass. The Schwarzschild metric is the spherically symmetric vacuum solution of Einstein's field equations. That is, it is only valid in the empty space outside the body.

Now consider what happens when $r = 2G/Mc^2$. Then

$$ds^2 = (1-2G/Mc^2\,/2G/Mc^2)dt^2 - (1-2G/Mc^2\,/2G/Mc^2)^{-1}\,dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\varphi^2$$

$$ds^2 = (1-1)dt^2 - (1-1)^{-1}\,dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\varphi^2$$

$$ds^2 = 0\,dt^2 - dr^2/0 - r^2d\theta^2 - r^2\sin^2\theta d\varphi^2$$

Hence, this value of r results in a singularity and is known as the critical or ***Schwarzschild radius***, $r_c = 2G/Mc^2$. The coefficient of dt^2 goes to zero, which means that O would see O's clock appear to run infinitely slowly. A message emitted at some time, to, would not arrive at a larger value r until an infinite time later. In fact, signals emitted at $r < r_c$ never come out of this region of S-T. A massive body completely enclosed within r_c could not radiate out into the rest of the universe and would be appear to be invisible. For $r < r_c$, the sign on the r term reverses. The critical radius corresponds to a surface of infinite red shift. Black holes are detectable through the gravitational and electromagnetic fields they would set up, but not by radiation they emit.

Derivation of Schwarzschild Metric

Consider two inertial frames, O' and O, both of which are freely falling, in a centrally symmetric gravitational potential Φ. An observer in O' is at distance r from the central mass distribution, M, and has just started to accelerate radially towards M but has velocity is zero. Since O' is freely falling, his differential line element in four-space is:

$$ds^2 = c^2dt'^2 - r'^2(\sin^2\theta'\, d\phi'^2 + d\theta'^2) - dr'^2.$$

This is nothing more than the Pythagorean expression for the differential separation of two points or events in a four-dimensional flat space expressed in spherical coordinates.

Now consider another observer in frame O that is at a sufficiently large distance r from M so that his potential is nearly zero but has escape velocity. Hence, $1/2V^2 = \Phi$ at all times. When O passes O' at distance r, he is coasting at velocity V relative to O'. Since both observers are in inertial frames, we may use the Lorentz transformation equations to deduce what O's line element would be as seen by O. Hence,

$$dt = dt'/(1- v^2/c^2)^{1/2} \text{ and } dr = dr'(1- v^2/c^2)^{1/2}$$

But $v^2 = V^2 = 2\Phi$, so

$$dt = dt'/(1- 2\Phi/c^2)^{1/2} \text{ and } dr = dr'(1- 2\Phi/c^2)^{1/2}$$

Now we may write

$$ds^2 = c^2dt^2(1- 2\Phi/c^2) - r^2(\sin^2\theta\, d\phi^2 + d\theta^2) - dr^2/(1- 2\Phi/c^2).$$

Here $r'^2(\sin^2\theta'\, d\phi'^2 + d\theta'^2) = r^2(\sin^2\theta\, d\phi^2 + d\theta^2)$, since there is no motion except along r and hence no angular momentum. This represents a translation of the clock rate and scale length in O's frame as observed by O. For the gravitational field of M, $\Phi = GM/r$. Therefore:

$$ds^2 = c^2 (1- 2GM/rc^2)dt^2 - r^2(\sin^2\theta\, d\phi^2 + d\theta^2) - dr^2/(1- 2GM/rc^2).$$

Such a line element is called a ***Schwarzschild metric***. Now when $r_s = 2MG/c^2$, we note that something odd takes place. This is the Schwarzschild radius.

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