4-5.1 Types of Broadening:

4-5.1A Natural Broadening:

This results from the uncertainty principle. If an electron spends time Δt in a given level, then its energy E is unknown by an amount

$$\Delta E = h/2\pi\Delta t \tag{4-71}$$

If E_o is the mean energy level corresponding to λ_o, then the level is widened by $2\Delta E$. That is:

$$E = E_o \pm \Delta E. \tag{4-72}$$

Then the broadening function is given by quantum mechanics to be:

$$\phi_\nu\, d\nu = \gamma_N d\nu \frac{1}{4\pi^2(\nu_0 - \nu)^2 + (\gamma_N/2)^2} \tag{4-73}$$

Here γ_N is the natural damping constant = $\gamma_1 + \gamma_2$ and has dimensions of frequency. The denominator in (4-73) has dimensions of frequency squared or sec^{-2}. Thus ϕ_ν has dimension of reciprocal frequency or sec. The γ-values are the reciprocals of the mean lifetimes of the 2 states involved. The usual value for Δt is 10^{-8} sec., which results in a natural width of about 0.0004 Å. In general

$$\gamma_N = 8\pi^2 e^2/3m_e\lambda^2 c = 0.223/\lambda^2\ sec^{-1}, \tag{4-74}$$

where λ is in cm.

Now there exist what are called metastable states. These are excited levels with small values for *f* and therefore long values of Δt. Hence ΔE is small and the resulting line is very narrow.

4-5.1B Doppler Broadening:

This results from the motions of atoms causing Doppler changes in the central wavelength of the line produced by a particular atom. The broadening function is found by combining the Doppler equation with the Boltzmann speed distribution function. This results in the following:

$$\phi_\nu\, d\nu = (1/\sqrt{\pi})(d\nu/\Delta\nu_D)\exp\{-[(\nu-\nu_o)/\Delta\nu_D]^2\} \tag{4-75}$$

where $\Delta\nu_D$ or $\Delta\lambda_D$ is the Doppler width. This corresponds to the frequency or wavelength shift that results from the most probable speed of the atoms, v_p.

$$v_p = [2kT/m]^{1/2}, \tag{4-76}$$

where m is the mass of the atom. The Doppler width is not a FWHM, but just a width corresponding to v_p to either side of the central rest wavelength, λ_o. Note that ϕ_ν has dimensions of 1/frequency as in (4-73) for natural broadening.

Now from the Doppler relation

$$\frac{\Delta\lambda_D}{\lambda_0} = \frac{v_p}{c} \qquad (4\text{-}77)$$

and using (4-76) we get:

$$\Delta\lambda_D = \lambda_o v_p/c = (\lambda_o/c)\sqrt{2kT/m} \qquad (4\text{-}78)$$

If there is turbulence, we have additional turbulent velocity v_T. Then the Doppler width becomes

$$\Delta\lambda_D = \frac{\lambda_0}{c}\left[\frac{2kT}{m} + v_T^2\right]^{1/2} \qquad (4\text{-}79)$$

If in addition, the star has significant rotation, then we need to add another term inside the square bracket in (4-79). Do RJP-92 and 94.

4-5.1C Pressure, Collisional, or Stark Broadening

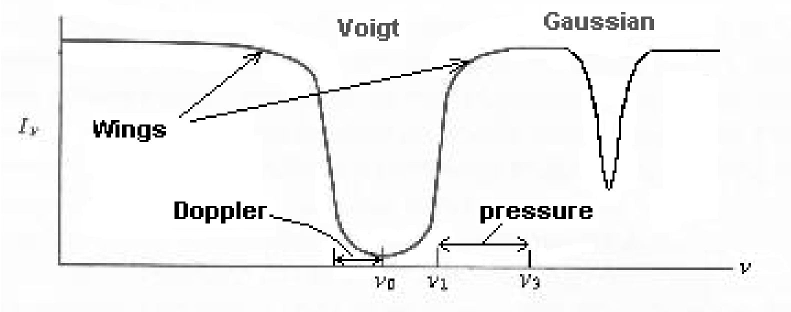
Pressure broadening is extremely complicated and in practice some form of approximation theory is used. In essence, broadening is the result atomic interactions, either due to collisions or atoms coming sufficient close to one another so that the energy states of the outer electrons are perturbed. Therefore when electron transitions occur, the energy states are blurred over a range of energies rather than a uniquely defined one. It is essentially pressure broadening that causes a continuous spectrum to be radiated according to Kirchhoff's 1st Law. In fact, an impressive experiment to do is to take a discharge tube of some gas, such H or He, and gradually increase the pressure in the tube while passing an electric current under high voltage through the gas to produce an emission spectrum. One observes each of the narrow emission lines gradually broaden until they merge into one another to form a continuous spectrum. This is essentially the kind of experiments that Kirchhoff and Bunsen carried out.

It turns out that most atoms have a pressure broadening function that is of the same form as the natural broadening function but with its own characteristic damping constant, $\gamma_P = t/2$, where t is the mean time between encounters and hence γ_P is essentially the frequency of disturbing collisions or encounters. It can be shown that the pressure broadening function may be approximated as

$$\phi_\nu = \frac{1}{2\pi^2 t}\frac{1}{(\nu-\nu_0)^2 + (1/2\pi t)^2} \qquad (4\text{-}80)$$

An expression for t may be found by applying statistical kinetic theory but this is too complex to present here. Again, ϕ_ν has units of 1/frequency.

4-5.1D Total Broadening Function



The total broadening function is a combination of all of the above mechanisms. For main sequence stars, the broadening function is usually dominated by rotation, whereas for supergiants it is Stark broadening that dominates. This is because supergiants are slow rotators. Giant stars are a case in between. In any event, the combined broadening function gives rise to a profile that is not Gaussian and is called a Voigt profile. Examples of the two different profiles are shown in the schematic diagram above.

In the Voigt profile, the frequency ν_3 indicates where the absorption in the line ends and the value of I_ν is at the continuum level. Pressure broadening dominates in those regions referred to as the wings of the profile, for example between ν_1 and ν_3. Doppler broadening dominates near the line center. What about natural broadening? The general rule for the where the transition from pressure broadening to pressure broadening occurs in the profile is $3\Delta\nu_D$ from the line center, ν_o. Some authors say $1.8\Delta\nu_D$. So this matter is equivocal. One could use a mean value of $2.4\Delta\nu_D$, as is done below. A Gaussian profile is what you get when doing problem No. 80.

4-6. Abundance Determination; Curve of Growth.

The strength of a spectral line depends on temperature vis-à-vis the Boltzmann and Saha equations and also on the abundance of the element producing the line through the number density and in turn, the optical depth. In order to sort out these two parameters in terms of how each affects the strength of a line, one constructs theoretical values for the equivalent width of a line in terms of these parameters. The method is called Curve of Growth. The equivalent width of a line is defined as:

$$W_\lambda = \int_\lambda (I_c - I_\lambda)/I_c \, d\lambda = \int_\lambda (1 - I_\lambda/I_c) \, d\lambda \qquad (4\text{–}81)$$

The units of W are cm or Ångstroms. To simplify matters, assume $j_\lambda = 0$ or is negligible. Then $I_\lambda = I_o e^{-\tau}$, which we substitute into the above. This gives us:

$$W_\lambda = \int_\lambda (1 - I_o e^{-\tau}/I_c) \, d\lambda. \qquad (4\text{-}82)$$

But, we recognize that for a particular line $I_o = I_c$, that is the continuum is constant. Hence:

$$W_\lambda = \int_\lambda (1 - e^{-\tau}) \, d\lambda., \qquad (4\text{–}83)$$

where τ is a function of wavelength through (4-69).

Now we introduce column density, N_a, to be the total number of absorbing atoms in a tube or column, of unit cross-section, extending through the atmosphere of the star along the line of sight to the star. The units for N_a are number per cm^2. We can imagine N_a to be all the atoms in such a tube that participate in producing the absorption feature. Hence, $N_a = \int n \, dz$. Also, σ_λ does not depend on z, but only the atom causing the absorption. Then for an atmosphere of some arbitrary thickness, (4-26) yields:

$$\tau_\lambda = \int \alpha_\lambda \, dz = \int n\sigma_\lambda \, dz = N_a \sigma_\lambda \qquad (4\text{-}84)$$

By expressing τ in terms of N_a, we do not have to concern ourselves with the way the volumetric number density of atoms, n, varies with z. Now using the expression for σ_ν from (4-69) and converting to wavelength units, we have:

$$\tau_\lambda = N_a\sigma_o\,\phi_\lambda = N_a[1-\exp(-hc/\lambda_o kT)](\pi e^2/m_e c^2)\lambda_o^2 f\,\phi_\lambda. \qquad (4\text{–}85)$$

where $\sigma_o = [1-\exp(-hc/\lambda_o kT)](\pi e^2/m_e c^2)\lambda_o^2 f$. Now (4-83) may be evaluated numerically for the following cases:

If $|(\lambda - \lambda_o/\Delta\lambda_D| < 2.4$, then ϕ_λ is given by the Doppler function.

If $|(\lambda - \lambda_o/\Delta\lambda_D| > 2.4$, ϕ_λ is given by the pressure broadening function.

As an example, consider a weak line so that $\tau_\lambda = N_a\sigma_o\phi_\lambda \ll 1$ for all λ. Then $e^{-\tau}$ in (4-83) may be expanded in a series such that $e^{-\tau} = 1 - \tau$. Then:

$$W = \int_\lambda (1 - e^{-\tau})\, d\lambda = \int_\lambda [1 - (1 - \tau)]\, d\lambda = \int_\lambda \tau\, d\lambda = N_a\sigma_o \int \phi_\lambda\, d\lambda = N_a\sigma_o, \qquad (4\text{-}86)$$

here the integral is over all wavelengths and we have used (4-70) for the value of the integral of the broadening function. In this case, the line is said to be optically thin and the profile is Gaussian in appearance and shallow. This is schematically illustrated in Fig. 4 as the line feature labeled A.

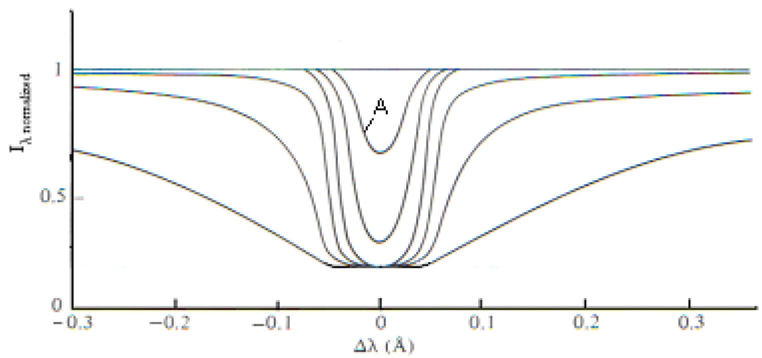


Fig. 4 Change in line profile resulting from increasing density

In a curve of growth diagram, log W is usually plotted versus log $(N_a\sigma_o)$. See Fig. 5, which is an example of a curve of growth. At low density or optical depth, the curve is linear and labeled N in Fig. 5. This corresponds to the result above in (4-86). As the column density increases, the line gets deeper and wider. Eventually the center of the line becomes optically thick as the maximum amount of flux at the center of the line is absorbed. As the density increases more, the bottom of the line flattens out and the slope of the curve of growth turns over. Now log W goes as $(\ln N_a\sigma_o)^{0.5}$. Further increases in the density increases the width of the pressure broadening profile and the wings of the line begin to deepen. Then Log W increases more rapidly as $(N_a\sigma_o)^{0.5}$. Notice in Figure 4 that even when the line is saturated, the flux at the bottom of the line does not go to zero. There is always some flux surviving from the higher layers of the atmosphere, where $\tau < 2/3$.

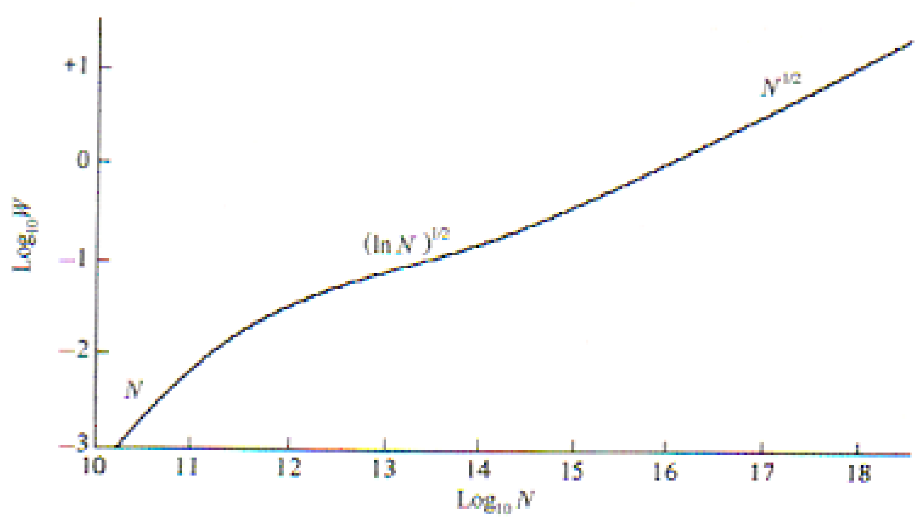


Fig. 5. A curve of growth diagram. Here $N = N_a\sigma_o$ The changing functional dependence of W on N is denoted along the curve.

We now present an example of how one uses the curve of growth to determine the abundance of an element. One needs to measure the equivalent widths of one or more spectral lines corresponding to a transition from some particular energy level in the atom. In practice, this in itself is not easy, but it can be done. One uses more than 1 line in order to gain statistical advantage.

In the Sun, there are Na I lines at 3032.4Å and 5890.0Å. Both lines are produced by a transition from the ground state to a higher state and so both lines correspond to the same value of N_a. The measured value of W for the first line is 0.088Å and W=0.730Å for the second line. These transitions have f values of 0.0214 and 0.645 respectively. A generic curve of growth for any line has been computed for T=5800K and P_e=10dyne/cm^2 by L. H. Aller, for values of $\log(W/\lambda)$ versus $\log(N_a f\lambda/5000)$. Of course computing such a curve is the crux of the method and difficult to do theoretically. When done, such a calculation must be considered a major accomplishment.

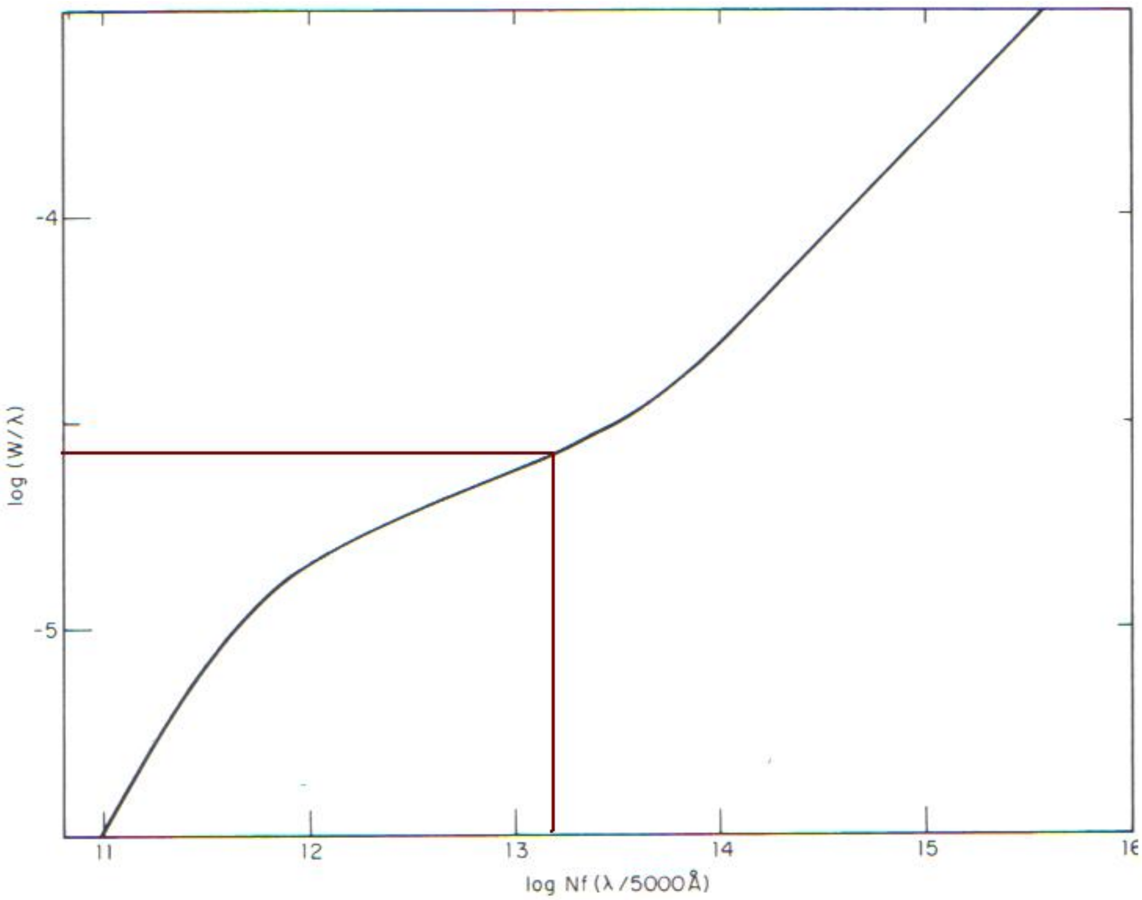
Now $\log(W/\lambda)$ for the first line is –4.58 and for the second line –3.90. Using Aller's curve of growth, shown below, it is found that $\log(N_a f\lambda/5000)$ for the first line is 13.20 and 14.83 for the second line. We solve for N_a by also computing $\log(f\lambda/5000)$ using the values of f given above. This then leads to

$\text{Log } N_a = \log(N_a f\lambda/5000) - \log(f\lambda/5000) = 13.20 - (-1.85) = 15.05$ for the line at 3302.4Å, and

$\text{Log } N_a = \log(N_a f\lambda/5000) - \log(f\lambda/5000) = 14.83 - (-0.12) = 14.95$ for the line at 5890.0Å.

The average value of Log N_a = 15.00. This means there are 10^{15} sodium atoms in the ground state per cm^2 in the atmosphere of the Sun. If we had used more lines, we could obtain a statistically better result.

To find the total number of sodium atoms per cm^2, we use the Boltzmann and Saha equations. The difference in energy between the two transition states, ΔE, in the Boltzmann equation is just the energy of the emitted photons given by hc/λ for the wavelengths given above. The exponential terms in the Boltzmann equation are then 5.45×10^{-4} for the 1st line and 1.48×10^{-2} for the 2nd line. These small values indicate that nearly all the Na I atoms are in the ground state, j=1 So we may take $N_{Ij} = N_{I1} = 10^{15}$ and use the Saha equation to find the ratio between N_{II} and N_I, with $B_{II} = 1$, $B_I = 2.4$, and the ionization potential, X_I =5.14ev. We may then find the total value of N_a for sodium.



A general curve of growth for the Sun. (Figure from Aller, *Atoms, Stars, and Nebulae*, Revised Edition, Harvard University Press, Cambridge, MA, 1971.)

Do RJP-101, 103, and 105.