CHAPTER 4B

4-4 Opacity

We now turn to the matter of determining what contributes to the optical depth of a medium, such as the layers of a star. The theory here can be quite complex and a thorough exposition would take us deep into quantum mechanics. So we can not completely go there.

Previously we have introduced the absorption coefficient α, which is the fraction of the intensity that is absorbed over a distance, dx. This parameter has dimensions of cm^{-1}. In general α depends on wavelength or frequency and is a function of temperature, pressure, and chemical composition. We now introduce what is called the mass absorption coefficient, κ, which has units of cm^2/g and is also a function of wavelength. The relation between these two parameters is $\alpha = \kappa\rho = n\sigma$, where ρ is the mass density in g/cm^3, n is the number density of absorbers with cross-section, σ, for absorption or scattering Again, in general, these parameters are wavelength dependent. The term opacity in general refers to the degree that a medium is opaque to radiation. Therefore, it can refer to τ, α, or κ, but is usually meant to refer to κ by most authors.

4-4.1 Classical Theory.

Radiation is absorbed by electrons which may be free or bound in an atom and occupying some energy level. The classical picture of an electron bound to its nucleus was already introduced in the derivation of Planck's Law.   In this case, the electron is considered a harmonic oscillator responding to a variable electromagnetic field, which is what radiation is.   The electron is set into motion by this field and in so doing energy is removed from the field. However, once the electron is set into motion, it begins to radiate and rid itself of some of the energy it has absorbed. In a classical oscillator, this is called the damping response of the electron. That is, its motion is damped by the fact that it is losing energy by radiating. If the motion were not damped, the amplitude of the electron would continue to grow larger and larger. By radiating, the electron obtains an equilibrium condition for its oscillation. That is, we consider the electron as a vibrating dipole carrying out damped harmonic motion and radiating in all directions. In this sense it is a Hertzian oscillator. We now can set up the equation of motion of the electron using Newton's 2nd Law of motion for this equilibrium condition and then attempt to solve the differential equation.

$$F_{net} = ma = \Sigma \, F_i \tag{4-32}$$

$$ma = m\ddot{x} = -Kx - g\dot{x} + E_x e \tag{4-33}$$

Note that each term in this equation is a force. Here we are considering the vibration of the electron along the x-axis, m is the mass of the electron, K is the elastic restoring force produced by the nucleus, e is the charge on the electron, and E is the electric field vector of the electromagnetic wave. The term $g\dot{x}$ is the damping force, which like the elastic restoring force is negative because it is opposite the electric force of the electric field of the radiation. We now rewrite (4-33) as

$$\ddot{x} + \gamma\dot{x} + \omega_o^2 x = E_x e/m \tag{4-34}$$

Here $\gamma = g/m$ is the damping constant, and $\omega_o^2 = K/m = (2\pi\nu_o)^2 = 4\pi^2\nu_o^2$. Now the electric field of the radiation is time dependent and may be written as $E_x = E_{ox}\exp(-i\omega t) = E_{ox}\exp(-i2\pi\nu t)$. The equation

(4-34) is a classical differential equation whose solution is $x = x_o\exp(-i\omega t)$. By substituting these latter two expressions back into (4-34), we get

$$(-\omega^2 + i\gamma\omega + \omega_o^2)x_o = eE_{ox}/m \tag{4-35}$$

From this we can find the value of x_o:

$$x_o = eE_{ox}/[m(-\omega^2 + i\gamma\omega + \omega_o^2)]$$

Then substitute this value for x_o back into our general solution for x and we get:

$$x = x_o\exp(-i\omega t) = eE_{ox}\exp(-i\omega t) / [m(-\omega^2 + i\gamma\omega + \omega_o^2)] \tag{4–36}$$

or substituting for ω in terms of frequency:

$$x = eE_{ox}\exp(-i2\pi\nu t) / [m(-4\pi^2\nu^2 + i\gamma2\pi\nu + 4\pi^2\nu_o^2)] \tag{4–37}$$

For astrophysical applications, we are interested in the propagation of electromagnetic radiation through gases, which are really **dielectrics**. Electric fields applied to a dielectric cause polarization of the medium in that the electrons in atoms are displaced relative to the positive charges in the nucleus. Remember that electromagnetic radiation is a time dependent electric field at a any point in the dielectric or for any atom. The displacement of the electrons from their unperturbed positions relative to the nucleus, x, gives rise to what is called a dipole moment, p, defined as:

$$p = ex = \alpha E \tag{4-38}$$

Here E is the electric field either static or time dependent and α is called the susceptibility or polarizability of the medium; it is not the absorptivity. The above leads to

$$\alpha = ex/E \tag{4-39}$$

And for a time dependent electric field we get;

$$\alpha = ex/E_{ox}e^{-i\omega t} \tag{4-40}$$

So we have a time dependent dipole or harmonic oscillator and the solution for the position of the charge is given by (4-37). Substituting for x in (4-40) from (4-37), the factor $e^{-i\omega t}$ in the numerator and denominator cancel yielding:

$$\alpha = e^2/ [4\pi^2 m(\nu_o^2 - \nu^2 + i\gamma\nu/2\pi)] \tag{4–41}$$

We see that α depends only on the properties of the atom and the frequency and not the electromagnetic field in which the atom finds itself.

If there are N_e electrons per cm^3, then according electromagnetic theory, the dipoles will contribute a term $4\pi N_e p$ to the total electric field in the medium, D. Then

$$D = E + 4\pi N_e p = E + 4\pi N_e \alpha E = E(1 + 4\pi N_e \alpha) \tag{4-42}$$

Therefore, $D = \varepsilon E$, where ε is the electric permittivity of the material, defined as:

$$\varepsilon = 1 + 4\pi N_e \alpha \qquad (4\text{--}43)$$

Substitute here for α from (4-41), and we have

$$\varepsilon = 1 + N_e e^2 / \, [\pi m (v_o^2 - v^2 + i\gamma v/2\pi \,)] \qquad (4\text{--}44)$$

In any medium, the index of refraction, n, is c/v and $n = \varepsilon^{1/2}$. We see from (4-44), that ε is complex and hence n is complex in general. A complex index of refraction is needed when radiation passing through a medium is absorbed. The complex part of the index of refraction is related to the absorption coefficient σ. It is this relation that we want to determine. To do this, let us write n as a complex number:

$$n = \varepsilon^{1/2} = n - i\kappa. \qquad (4\text{-}45)$$

Here n is the ordinary index of refraction when there is no absorption and κ is the absorptivity parameter. Using (4-44), expand the square root of ε by the binomial theorem, $(1+x)^n = 1 + nx + \dots +$, where $n = 1/2$ and x is the 2nd term on the right in (4-44). We then get:

$$\varepsilon^{1/2} = 1 + (1/2)Ne^2 / \, [\pi m (v_o^2 - v^2 + i\gamma v/2\pi \,)] + \dots \qquad (4\text{--}46)$$

We transform (4-46) into a form more similar to (4-45) by multiplying the numerator and denominator of the 2nd term on the right by the complex conjugate of the denominator. Then

$$n - i\kappa = 1 + (Ne^2/2\pi m) \, \frac{1}{v_o^2 - v^2 + i\gamma v/2\pi} \left[\frac{v_0^2 - v^2 - i\gamma v/2\pi}{v_0^2 - v^2 - i\gamma v/2\pi} \right] \qquad (4\text{-}47)$$

$$n - i\kappa = 1 + (Ne^2/2\pi m) \left[\frac{v_0^2 - v^2}{\left(v_0^2 - v^2\right)^2 + \left(\gamma/2\pi\right)^2 v^2} - \frac{i\gamma v/2\pi}{\left(v_0^2 - v^2\right)^2 + \left(\gamma/2\pi\right)^2 v^2} \right] \qquad (4\text{-}48)$$

Now n is the real part of (4-48), viz.,

$$n = 1 + (Ne^2/2\pi m) \frac{v_0^2 - v^2}{\left(v_0^2 - v^2\right)^2 + \left(\gamma/2\pi\right)^2 v^2} \qquad (4\text{-}49)$$

and κ is the imaginary part:

$$\kappa = (Ne^2/2\pi m) \frac{\gamma v/2\pi}{\left(v_0^2 - v^2\right)^2 + \left(\gamma/2\pi\right)^2 v^2} \qquad (4\text{-}50)$$

For most astrophysical applications, we are interested in values of v near the resonant frequency v_o. So when $v \cong v_o$ we may write $v_o^2 - v^2 = 2v(v_o - v)$. Then (4-50) becomes:

$$\kappa = (Ne^2/2\pi m) \frac{\gamma\nu/2\pi}{4\nu^2(\nu_0 - \nu)^2 + (\gamma/2\pi)^2\nu^2}$$

$$\kappa = (Ne^2/8\pi m\nu) \frac{\gamma/2\pi}{(\nu_0 - \nu)^2 + (\gamma/2\pi)^2} \tag{4-51}$$

$$\kappa = (Ne^2\gamma/16\pi^2 m\nu) \frac{1}{(\nu_0 - \nu)^2 + (\gamma/2\pi)^2} \tag{4-52}$$

What we need to do now is relate κ to the absorption coefficient per atom σ. To do this we need the expression for a traveling electromagnetic wave, viz.,

$$E = E_o \exp[2\pi i\nu(t - x/v)] \tag{4-53}$$

This represents an electromagnetic wave traveling in the x-direction with speed v, where $v = c/\mathsf{n}$. Substituting $\mathsf{n} = n - i\kappa$ into (4-53) we get

$$E = E_o \exp[2\pi i\nu(t - xn/c + xi\kappa/c)] \tag{4-54}$$

$$E = E_o \exp[2\pi i\nu(t - xn/c]\exp(- 2\pi\nu\kappa x/c) \tag{4-55}$$

This represents a wave undergoing an attenuation by the factor $e^{-\beta x}$, where $\beta = 2\pi\nu\kappa/c$. Now the intensity of radiation varies with the square of the amplitude of the electromagnetic wave, that is, $I \sim E^2$, so

$$I = I_o e^{-2\beta x} = I_o \exp(-4\pi\nu\kappa x/c) \tag{4-56}$$

But we have seen before that $I = I_o e^{-\kappa\rho x}$. Hence, comparing the exponential arguments we have:

$$\kappa_\nu\rho = \alpha_\nu = \sigma_\nu n = 4\pi\nu\kappa_\nu/c, \tag{4-57}$$

Recall that κ_ν is called the mass absorption coefficient and ρ is the mass density. Substituting for κ from (4-52) we get:

$$\kappa_\nu\rho = \sigma_\nu n = (N_o e^2\gamma/4\pi mc) \frac{1}{(\nu_0 - \nu)^2 + (\gamma/2\pi)^2} \tag{4-58}$$

Here N_o is the number of bound electrons with resonant frequency ν_o per unit volume and n is the number of atoms per unit volume possessing such bound electrons, so they are really the same number.

Therefore, $$\sigma_\nu = (e^2\gamma/4\pi mc) \frac{1}{(\nu_0 - \nu)^2 + (\gamma/2\pi)^2} \tag{4-59}$$

Quantum theory leads to the same result with a modified meaning of γ and N_o replaced by $N_o f$, where f is called the oscillator strength or probability of the transition. The values of f for some of the members of the hydrogen Balmer series are:

$f_{H\alpha} = 0.6408$ $\qquad\qquad$ $f_{H\beta} = 0.1193$ $\qquad\qquad$ $f_{H\gamma} = 0.0447$ $\qquad\qquad$ $f_{H\delta} = 0.0221.$

For the strong lines of the alkali atoms, $f \cong 0.98$.

Now recall that σ_ν is the absorption coefficient per atom at frequency ν, and from (4-58):

$$\sigma_\nu = \kappa_\nu \rho / n \tag{4-60}$$

The total amount of isotropic radiation absorbed per unit volume in a unit solid angle per second in the range $d\nu$ will be

$$dI_\nu = n\sigma_\nu I_\nu d\nu d\omega \tag{4-61}$$

The total amount of absorbed radiation per second over all solid angles per cm^3 is then:

$$\Delta I_\nu = n \int \int \sigma_\nu I_\nu \, d\nu \, d\omega = 4\pi n \int \sigma_\nu I_\nu \, d\nu \tag{4-62}$$

If we consider the continuum radiation in the vicinity of a spectral line, we may assume that I_ν does not vary much over the frequency interval of integration where σ_ν is significant. Then I_ν may be taken outside of the integral and we have

$$\Delta I_\nu = 4\pi n \, I_\nu \int \sigma_\nu \, d\nu, \tag{4-63}$$

where the expression for σ_ν is taken from (4-59) and we include the oscillator strength. This integral may be evaluated by trigonometric substitution so that (4-63) becomes

$$\Delta I_{\nu o} = 4\pi n \, I \, (\pi e^2 / mc) f \tag{4-64}$$

We must recognize that this equation has limited application. It has been assumed that I is independent of ν and that $\Delta I_{\nu o}$ is the total absorbed intensity over the entire spectrum as a result of electrons oscillating at resonant frequency ν_o. In other words, it is the total absorption in a specific absorption line or the strength of a line. It does not produce the details of the profile of an absorption line. The latter can only be found by integrating (4-62) in detail. We take this matter up later.
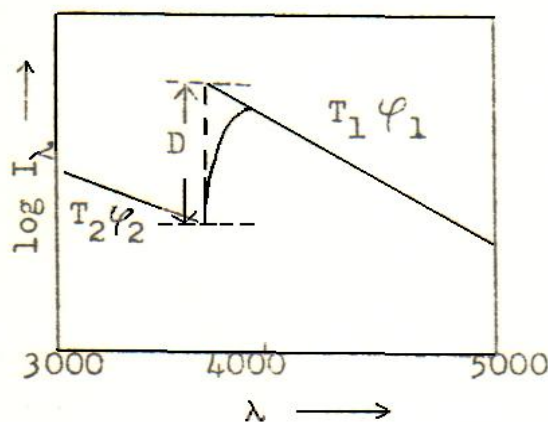
Definition of the Balmer
Discontinuity, D. The energy curve
longward of the Balmer limit is
characterized by the parameters
$T_1 \varphi_1$; the ultraviolet energy
curve by $T_2 \varphi_2$. The discontinui-
ty in I_λ at the Balmer limit is
called D.

4-4.2 Other Sources of Opacity

As we have seen, one of the limitations in determining an atmospheric model of a star is in calculating σ_λ, which in general is a function of temperature, electron pressure, and chemical composition and because values of f are uncertain.

It turns out that H and He, because of their high excitation potentials, are not important for spectral classes later than F but important for O, B, and A star. The Balmer discontinuity may be used as a measure of the importance of H absorption. The Balmer discontinuity arises from the Balmer continuum absorption that is short ward of the Balmer limit. This is

illustrated in the diagram to the left. The Balmer discontinuity, D, is strongest at spectral type A2.

The spectra of two stars, illustrating how the Balmer discontinuity changes with spectral type is shown in the diagram below, where wavelength increases to the right. The spectrum for the O9 V star
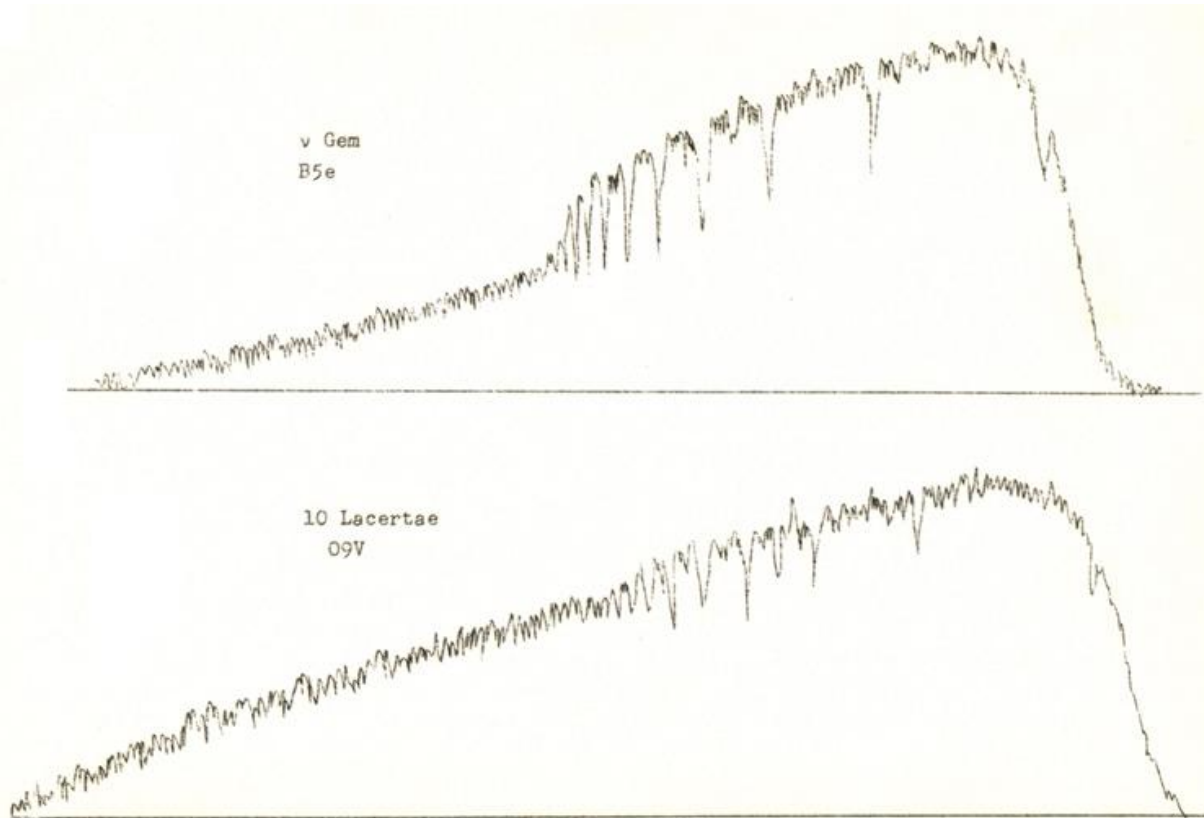
shows no noticeable value for D, but there is a conspicuous value of the Balmer discontinuity for the later type star of spectral type B5e. The letter "e" here means there are emission lines in the spectrum, implying the star has an extended atmosphere. One also notes the relatively strong Balmer absorption lines in the latter star's spectrum.

One should be aware that the spectra displayed in the above diagram reflect the convolution of the spectral response of the detector with the actual spectral distributions of the stars. This is why the spectrum falls off so sharply at the longer wavelengths and also in the ultraviolet, but more slowly. That is, the detector is more sensitive to the visible than it is at other wavelengths. The actual peak in the intensity for each curve is not in the red, as it appears, but in the UV.

In 1939, it was discovered that H⁻ made an important contribution to the opacity in the atmospheres of solar like stars. It turns out this opacity is gray, that is, it is independent of wavelength. In the 1920s, Sir Arthur Eddington, a renowned British astrophysicist, found a solution to the transfer equation for a gray atmosphere. This is:

$$T^4 = (3/4)T_e^{\,4}(\tau + 2/3)$$
(4-65)

T_e is the effective temperature of the star. One can see that this occurs in the atmosphere of the star at and optical depth of 2/3.

It worth repeating here what was stated previously, namely, that the radiation one observes coming from a star originates in various layers of different temperatures and not from just a very thin surface layer.   Since we observe the flux of a star to be the flux from layers of different temperatures, stars really do not radiate strictly as black bodies. The temperature of the photosphere determined from

Wien's Law or Planck's Law is only the effective temperature, T_e. Eddington's solution, gives the value of the temperature at various optical depths. Do RJP-76 & 80.

At very high temperatures, electron or Thomson scattering becomes important because of all of the ionization. The Thomson cross section is

$$\sigma_T = 8\pi e^4/3m_e^2c^4 = 6.65 \times 10^{-25} \text{ cm}^2. \qquad (4\text{-}66)$$

For a thorough analysis of the opacity one must find the total opacity due to all sources for both the continuum and the line. This is a very complex problem.

4.5 Details of Line Formation

For bound-bound transitions, we get discrete line absorptions or emissions as opposed to continuum absorptions and scatterings. From (4-59), the line absorption coefficient is

$$\sigma_v = (e^2\gamma/4\pi mc)\frac{1}{\left(v_0 - v\right)^2 + \left(\gamma/2\pi\right)^2} \qquad (4\text{-}67)$$

From quantum theory it is found that we need to include the oscillator strength, f, and a factor, in the square brackets, that takes into account induced emissions, which we shall not derive. Hence, σ_v becomes:

$$\sigma_v = [1\text{-}\exp(\text{-}hv_0/kT)](e^2\gamma/4\pi mc)\frac{1}{\left(v_0 - v\right)^2 + \left(\gamma/2\pi\right)^2}f \qquad (4\text{-}68)$$

If we integrate this over all frequencies we get:

$$\sigma_v = [1\text{-}\exp(\text{-}hv_0/kT)](\pi e^2/m_e c)\,f\,\phi_v \qquad (4\text{--}69)$$

which has units of cm^2 Hz, since we have integrated over all frequencies or wavelengths (Swihart, p. 131). Here ϕ_v is a broadening function. It should be noted that (4-69) is not valid if I_v is not constant over the entire spectrum.   If one wishes to know only the absorbed flux over a narrow range of frequencies or wavelengths near the resonant frequency v_o, one must integrate (4-62) over that particular range of frequencies. This is usually done by numerical methods.

The broadening function has the following property: $\phi_v\,dv$ or $\phi_\lambda\,d\lambda$ is the probability that the wavelength of the absorbed photon lies between λ and $\lambda + d\lambda$, assuming equal intensities for all λ. Hence,

$$\int_0^\infty \phi_v\,d_v = 1. \qquad (4\text{--}70)$$

The broadening function gives the line profile its shape, which is usually a Gaussian that has its peak at v_o, which is the central frequency of the line.