CHAPTER 4A

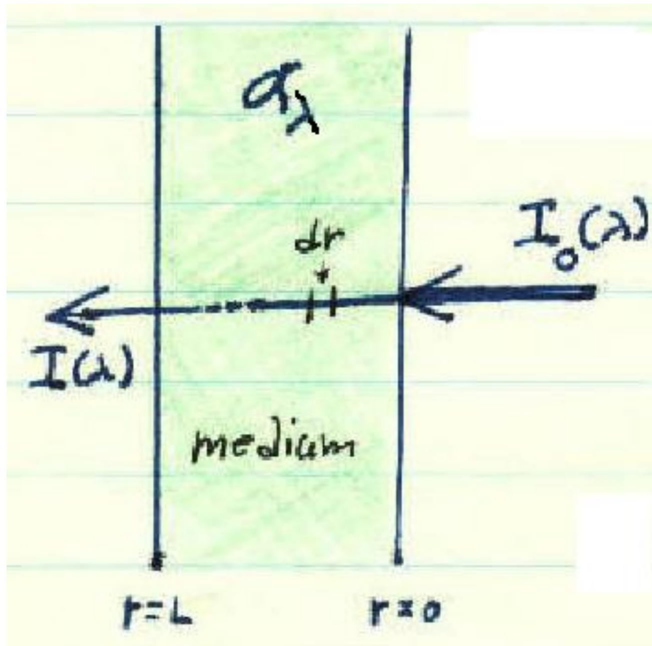
RADIATION TRANSFER AND LINE FORMATION

4-1. Emission and Absorption of Continuum Radiation

4-1A. The Transfer Equation

Consider a beam of radiation, $I_o(\lambda)$, incident on the boundary of a layer of an absorbing medium, as shown the diagram. The thickness of the layer is L. Let α_λ be the absorptivity of the medium at wavelength λ, that is the attenuation factor per unit path length through the medium. The absorptivity has dimensions of cm^{-1}, so $\alpha_\lambda dr$ is dimensionless. The differential change in the intensity of the beam at every point along the path through the layer is

$$dI_\lambda(r) = -\alpha_\lambda I_\lambda(r)dr \qquad (4\text{-}1)$$



The negative sign means that the change is a decrease in the intensity.

Now we introduce a quantity called the optical depth τ.

Let $d\tau_\lambda$ be the differential optical depth and τ_λ the total or integrated optical depth over some path. τ is dimensionless.

$$d\tau_\lambda = \alpha_\lambda dr \qquad (4\text{-}2)$$

and

$$\tau_\lambda = \int_{r=0}^{L} \alpha_\lambda dr = \int d\tau \qquad (4\text{-}3)$$

Then

$$dI_\lambda = -I_\lambda d\tau_\lambda \qquad (4\text{-}4)$$

or

$$\frac{dI_\lambda}{I_\lambda} = -d\tau_\lambda \qquad (4\text{-}5)$$

Integrating

$$\ln I_\lambda = -\tau_\lambda + C \qquad (4\text{-}6)$$

Determine C from boundary condition:

If $\tau_\lambda = 0$ then $I_\lambda = I_0(\lambda)$

Hence, $C = \ln I_0(\lambda)$

so $$\ln I_\lambda = -\tau_\lambda + \ln I_0(\lambda) \qquad (4\text{-}7)$$

$$\ln\left(\frac{I_\lambda}{I_0(\lambda)}\right) = -\tau_\lambda$$

$$\boxed{I_\lambda = I_0(\lambda) e^{-\tau_\lambda}} \qquad (4\text{-}8)$$

The above relation assumes that the medium does not add any radiation to the beam that emerges from it at $r=L$. If it does, we account for this by introducing an emissivity term $j_\lambda\, dr$, where j_λ is called the emissivity. The dimensions of j_λ are energy / unit vol / per sec / per Å / per steradian, then $j_\lambda\, dr$ has intensity units. Now (1) becomes

$$dI_\lambda = j_\lambda\, dr - \alpha_\lambda I_\lambda\, dr \qquad (4\text{-}9)$$

Transform (4-9) to $$dI_\lambda = j_\lambda \frac{\alpha_\lambda}{\alpha_\lambda} dr - \alpha_\lambda I_\lambda\, dr \qquad (4\text{-}10)$$

Since $\alpha_\lambda\, dr = d\tau_\lambda$: $$dI_\lambda = \frac{j_\lambda}{\alpha_\lambda} d\tau_\lambda - I_\lambda\, d\tau_\lambda \qquad (4\text{-}11)$$

Introduce "source function" $S_\lambda \equiv j_\lambda / \alpha_\lambda$

then (11) becomes: $dI_\lambda = S_\lambda d\tau_\lambda - I_\lambda d\tau_\lambda$ 4-(12)

or $$\frac{dI_\lambda}{d\tau_\lambda} = (S_\lambda - I_\lambda) \qquad 4\text{-}(13)$$

The above is the "EQUATION OF TRANSFER" in terms of τ_λ.

Solution for constant α_λ & j_λ:

Consider a uniform layer of thickness L. Assume j_λ and a_λ are independent of r. Then from

$$dI_\lambda + a_\lambda I_\lambda dr = j_\lambda dr \qquad (4\text{-}9)$$

$\cdot$//$\cdot$ by $e^{a_\lambda r}$: $e^{a_\lambda r}(dI_\lambda + a_\lambda I_\lambda dr) = j_\lambda e^{a_\lambda r} dr$ 4-(14)

The left side is recognized to be $d(e^{a_\lambda r} I_\lambda)$. Hence

$$d(e^{a_\lambda r} I_\lambda) = j_\lambda e^{a_\lambda r} dr \qquad 4\text{-}(15)$$

Now integrate over the layer from r=0 to r=L

$$\left[e^{a_\lambda r} I_\lambda\right]_0^L = \int_0^L j_\lambda e^{a_\lambda r} dr$$

or $$I_\lambda(L) e^{a_\lambda L} - I_\lambda(0) = \int_0^L j_\lambda e^{a_\lambda r} dr$$

$$I_\lambda(L) = I_\lambda(0) e^{-a_\lambda L} + \int_0^L j_\lambda e^{+a_\lambda r} dr \left[e^{-a_\lambda L}\right] \qquad (4\text{-}16)$$

For the last term $j_\lambda \int_0^L e^{+a_\lambda r} e^{-a_\lambda L} dr = j_\lambda e^{-a_\lambda L} \int_0^L e^{a_\lambda r} dr$

or, after integrating, we have:

$$j_\lambda e^{-\alpha_\lambda L}\left(\frac{1}{\alpha_\lambda}\right)e^{\alpha_\lambda r}\Big|_{r=0}^{L} = \frac{j_\lambda e^{-\alpha_\lambda L}}{\alpha_\lambda}\left(e^{\alpha_\lambda L}-1\right) \qquad 4\text{-}(17)$$

Putting (17) into (16) and simplify (17) we get:

$$I_\lambda(L) = I_\lambda(0)e^{-\alpha_\lambda L} + \frac{j_\lambda}{\alpha_\lambda}\left(1-e^{-\alpha_\lambda L}\right) \qquad 4\text{-}(18)$$

The 1st term on the right is the transmitted beam

The 2nd term is the emitted and/or scattered radiation from the layer. Consider two limiting cases of importance

a. $\alpha_\lambda L \ll 1$: Absorption is small; layer is essentially transparent or optically thin. Then from 4-(18)

$$I_\lambda(L) = I_\lambda(0) + \frac{j_\lambda}{\alpha_\lambda}\left(1-e^{-\alpha_\lambda L}\right)$$

expand $e^{-\alpha_\lambda L} = 1 - \alpha_\lambda L + \cdots$ Then

$$I_\lambda(L) = I_\lambda(0) + \frac{j_\lambda}{\alpha_\lambda}(1-1+\alpha_\lambda L)$$

$$I_\lambda(L) = I_\lambda(0) + j_\lambda L \qquad 4\text{-}(19)$$

b. $\alpha_\lambda L \gg 1$: $e^{-\alpha_\lambda L} \to 0$. So (18) becomes

$$I_\lambda(L) = j_\lambda / \alpha_\lambda = S_\lambda \qquad 4\text{-}(20)$$

Layer opaque and absorption large. The layer is said to be optically thick.

4.1B. The Local Thermodynamic Equilibrium Case

Consider an atmosphere to be made of various layers, each of which is in "local thermodynamic equilibrium" or LTE. LTE means that conditions do not vary from point to point within the layer. However, conditions do change from layer to layer.

Now consider a layer that is transparent but at $T>0$. This layer does not absorb but only radiates, obeying Planck's law. Therefore, the layer cools until it reaches $T=0$. Obviously this does not happen in a star. If the layer absorbed more than it emits, T would continuously increase. This would violate LTE assumption also. So, for a layer to be in LTE, it must absorb as much radiation as it emits and there is no net flow of radiation from the layer. Only in this way can T be constant. Hence, $I_\lambda = j_\lambda / \alpha_\lambda$ at every point. But in LTE,

$$I_\lambda = B_\lambda(T)$$

where $B_\lambda(T)$ is the Planck function. Hence,

$$j_\lambda = \alpha_\lambda B_\lambda(T) \qquad (4\text{-}21)$$

From (4-18) we get

$$I_\lambda(L) = I_\lambda(0)e^{-\alpha_\lambda L} + B_\lambda(T)\left[1-e^{-\alpha_\lambda L}\right] \qquad (4\text{-}22)$$

If there is no energy generation occuring in the layers, $L = 4\pi r^2 \sigma T^4$, where r is the radius of the layer, and T is its temperature, must be conserved from layer to layer. That is:

$$4\pi r_1^2 \sigma T_1^4 = 4\pi r_2^2 \sigma T_2^4$$

But let us take $r_2 > r_1$, then $T_2 < T_1$. That is, there must be a temperature gradient in the atmosphere such that

$$\frac{dT}{dr} < 0$$

Hence, for there to be a flow of radiant energy from the lower layers upward through the outers the temperature must decrease upwards through the star

Since T is the same within any given layer, there is no net flow from one place in a layer to another place in that same layer. The net flow is only from layer to layer in the direction of $-dT/dr$.

Now $$L_* = 4\pi R_*^2 \int_{\lambda=0}^{\infty} \int_{\Omega} I_\lambda(r,\theta) \cos\theta \, d\Omega \, d\lambda$$

$$L_* = 4\pi R_*^2 \sigma T^4$$

where $F_{bol}(R_*) = \sigma T^4$

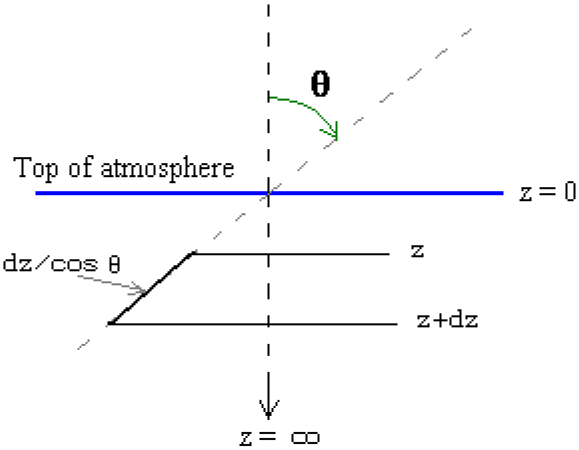
Since energy is conserved, in any layer of radius r

$$F_{bol}(r) = \sigma T_e^4 = \int_{\lambda} \int_{\Omega} I_\lambda(r,\theta) \cos\theta \, d\Omega \, d\lambda$$

where T_e is the effective temperature of the layer

4-2. Plane Parallel Atmosphere

If the atmosphere of a star is sufficiently thin, as is usually the case, the curvature can be ignored, and the atmosphere may be considered to be comprised of horizontal and parallel layers as shown below. The optical depth of the atmosphere to a depth z is



$$\tau_\lambda = \int_0^z \alpha_\lambda dz \qquad (4\text{-}26).$$

Now $e^{-\tau_\lambda}$ is the absorption factor for the material that is directly above the geometric depth z. See diagram. The top of the atmosphere is at z=0.

To simplify matters, we assume LTE. As we discussed above, this means conditions do not change from point to point along the path of integration. For example, the temperature is constant throughout the atmosphere and equal to some mean value that produces the observed continuum radiation. This is a first approximation. Furthermore, no incident radiation from the lower layers survives to emerge from the top of atmosphere. That is each layer of the atmosphere is optically thick to the radiation from below. If this were not so, there would be a net flow of radiation in violation of LTE.

Now integrate downwards into the atmosphere to z = ∞ (the bottom of the atmosphere) along the path at angle θ. Then the intensity emerging at an angle θ from a point at the top of the atmosphere is given by (4-16) with z= L-r as

$$I_\lambda(0,\theta) = \int_0^\infty j_\lambda e^{-\tau_\lambda(z)/\cos\theta} dz/cos\theta \qquad (4\text{-}27)$$

We saw that for the optically thick case, $I_\lambda(z,\theta) = j_\lambda/\alpha_\lambda$, for a layer at depth z. Also, in LTE, $I_\lambda(z,\theta) = B_\lambda(T)$, the Planck function, or $j_\lambda(z) = \alpha_\lambda(z)B_\lambda(T)$. Then (4-27) becomes

$$I_\lambda(0,\theta) = \int_{z=0}^\infty B_\lambda(T)e^{-\tau_\lambda/\cos\theta}\, \alpha_\lambda(z)dz/\cos\theta \qquad (4\text{-}28)$$

Now change variables to optical depth τ_λ, where $d\tau_\lambda = \alpha_\lambda(z)dz$

$$I_\lambda = \int_{\tau=0}^{\infty} B_\lambda(T)e^{-\tau_\lambda/\cos\theta}\, d\tau_\lambda/\cos\theta \qquad (4\text{–}29)$$

In (4-29), $T = T(\tau_\lambda)$ but cos θ is constant over the integration. Generally, T and τ_λ are not known functions of z.

Actually, the radiation that is observed comes from various layers of different temperatures and not from just a very thin surface layer. So what is called the photosphere of a star is of some extent and is defined as the layer down to an optical depth of 1. That is, we receive most of the observed radiation from a star down to where $I_{obs} = I_o e^{-\tau} = (1/e)I_o$ or $I_{obs} = 0.37I_o$, where I_o is the intensity at the bottom of the photosphere.

Since we observe the flux of a star to be the flux from layers of different temperatures, stars really do not radiate strictly as black bodies. The temperature of the photosphere determined from Wien's Law or Planck's Law is only an effective temperature.

4-3. General Solution of Transfer Equation

Now we consider a more general solution of the transfer equation, starting with (4-13)

$$dI_\lambda/d\tau_\lambda = S_\lambda - I_\lambda$$

For simplicity, we shall drop the wave length dependence and multiply both sides by $e^\tau d\tau$ to get:

$$dI\, e^\tau = S\, e^\tau d\tau - I\, e^\tau d\tau$$

or

$$dI\, e^\tau + I\, e^\tau d\tau = S\, e^\tau d\tau$$

Recognize that the left side is a differential and we get

$$\mathbf{d(e^\tau I) = S\, e^\tau d\tau}$$

Now integrate both sides from $-\tau$ to 0 :

$$Ie^0 - Ie^{-\tau} = \int_\tau Se^\tau d\tau\,,$$

or

$$I_e = I_i e^{-\tau} + \int_\tau Se^\tau d\tau, \qquad (4\text{-}30)$$

where I_e is the emergent intensity , that is, the intensity at z=0 (τ=0). I_i or I_o is the incident or original intensity entering the bottom of the layer from the layers below at higher optical depth. Remember that this is for a specific wavelength.

If we consider the special case where the source function is constant, we get

$$I_e = I_i e^{-\tau} + S \int_\tau e^\tau d\tau = I_i e^{-\tau} + S\,[e^0 - e^{-\tau}] = I_i e^{-\tau} + S\,[1 - e^{-\tau}], \qquad (4\text{-}31)$$

which is the same as (4-18)

Do RJP-70, 72, 74.