So

$$\bar{\epsilon} = -\frac{\partial}{\partial(1/kT)}\left[\ln \sum_n e^{-\epsilon_n/kT}\right] = \frac{h\nu}{2} + \frac{\partial}{\partial(1/kT)}[\ln x]$$

$$\bar{\epsilon} = \frac{h\nu}{2} + \frac{(\partial/\partial(1/kT))[1-e^{-h\nu/kT}]}{1-e^{-h\nu/kT}}$$

$$\bar{\epsilon} = \frac{h\nu}{2} + \frac{-\partial/\partial(1/kT)[e^{-h\nu/kT}]}{1-e^{-h\nu/kT}}$$

$$\bar{\epsilon} = \frac{h\nu}{2} + \frac{h\nu e^{-h\nu/kT}}{1-e^{-h\nu/kT}}$$

Multiply the second term in the above by $e^{h\nu/kT}/e^{h\nu/kT}$:

$$\bar{\epsilon} = \frac{h\nu}{2} + \frac{h\nu}{e^{h\nu/kT}-1} \tag{2-37}$$

As $T \to 0$, $h\nu/kT \to \infty$. So the 2nd term vanishes and $\bar{\epsilon} = h\nu/2$

So this term is the zero point energy.

When T is large, what happens. Expand e^x where $x = h\nu/kT$.

Now $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Then from (2-37)

$$\bar{\epsilon} = \frac{h\nu}{2} + h\nu\left[1 + \frac{h\nu}{kT} + \underbrace{\left(\frac{h\nu}{kT}\right)^2\left(\frac{1}{2!}\right) + \cdots - 1}_{\text{drop because } T^n \text{ is large when n >1}}\right]^{-1}$$

$$\bar{\epsilon} = \underbrace{\frac{h\nu}{2}}_{\simeq 0} + h\nu\left[\frac{h\nu}{kT}\right]^{-1}$$

$$\bar{\epsilon} = \frac{h\nu}{h\nu}(kT) \rightarrow kT$$ for very large T.

From electromagnetic theory it may be shown that

$$\bar{\epsilon} = \frac{\lambda^4}{8\pi} u_\lambda \qquad (2\text{-}41)$$

where $u_\lambda \, d\lambda$ = energy of radiation in the spectral interval $\lambda \rightarrow \lambda + d\lambda$

Now assume that T is sufficiently large that

$$\frac{h\nu}{2} \ll \frac{h\nu}{e^{h\nu/kT} - 1}$$

Then from (2-37) we may write

$$\bar{\varepsilon} = \frac{h\nu}{e^{h\nu/kT} - 1} \qquad (2\text{-}42)$$

Now combine (2-41) and (2-42) to obtain:

$$\frac{\lambda^4}{8\pi} u_\lambda = \frac{h\nu}{e^{h\nu/kT} - 1} \qquad (2\text{-}43)$$

And so,

$$u_\lambda = \frac{8\pi h\nu}{\lambda^4}\left(\frac{1}{e^{h\nu/kT} - 1}\right) \qquad (2\text{-}44)$$

It may be shown (Appendix C) that

$$u_\lambda = \frac{4}{c} F_\lambda \,, \tag{2-45}$$

so:

$$F_\lambda = \frac{2\pi h\nu c}{\lambda^4}\left[\frac{1}{e^{h\nu/kT}-1}\right] \tag{2-46}$$

Substituting $\nu = c/\lambda$

$$F_\lambda = \frac{2\pi hc^2}{\lambda^5}\left[\frac{1}{e^{hc/\lambda kT}-1}\right] \tag{2-47}$$

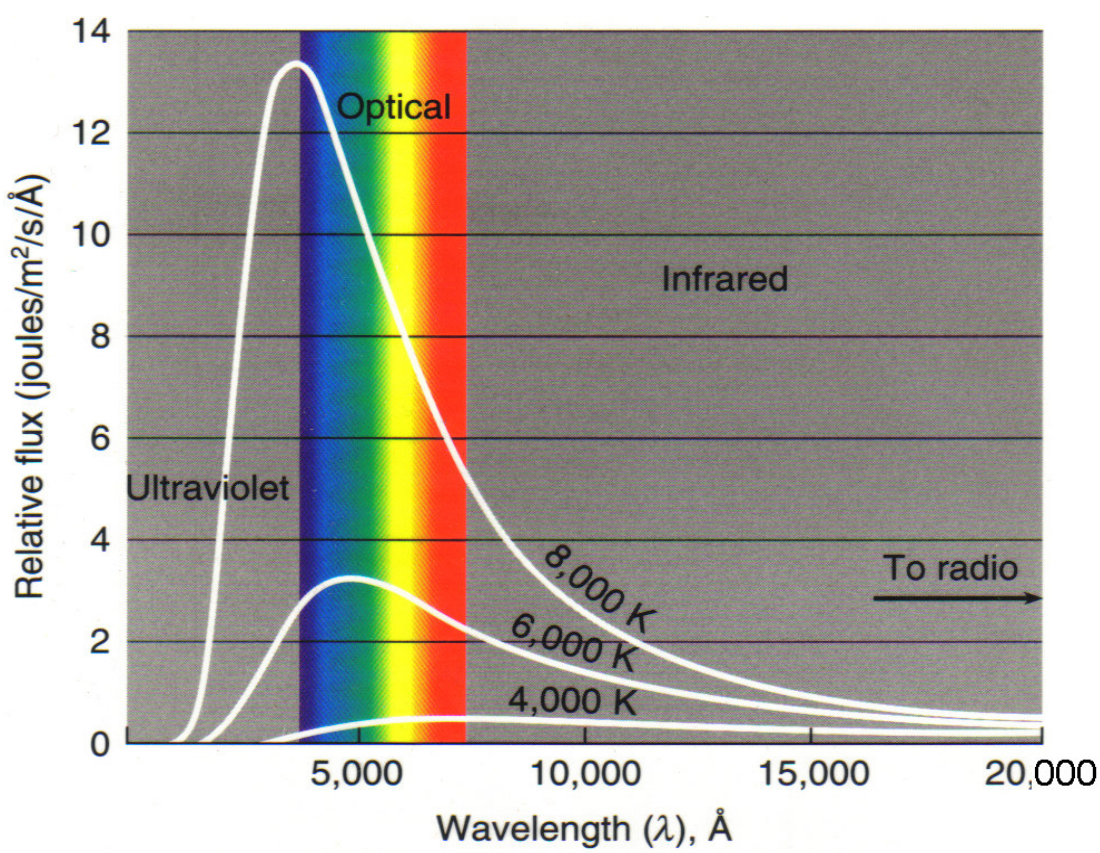
Equation (2-47) is Planck's radiation law. Let B_λ be the black-body intensity. Then for an isotropic and homogeneous radiation field, $F_\lambda = \pi B_\lambda$. Therefore:

$$B_\lambda = \frac{2hc^2}{\lambda^5}\left[\frac{1}{e^{hc/\lambda kT}-1}\right] \tag{2-48}$$

When B_λ is plotted versus wavelength for different temperatures, one gets the different curves that are shown in the diagram below. The curves are called "Black-body Curves" or "Planckians."

$$F_\lambda(T) = \pi B_\lambda(T) = \frac{\pi(1.191\times10^{-5})\lambda^{-5}}{(e^{1.439/\lambda T}-1)} \tag{2-49}$$

In this equation, λ must be in centimeters and the units for F_λ are ergs/cm^2/s/cm. **RJP-32, & 33.**



To find λ_{max}, the wavelength at which F_λ is a maximum, we differentiate Planck's law and set the result equal to zero. Solving for λ we get Wien's Displacement Law.

$$\lambda_{max} = \frac{0.2898}{T} \quad (\lambda \text{ in cm}) \tag{2-50}$$

For λ_{max} in Å:

$$\lambda_{max} = \frac{2.898 \times 10^7}{T} \tag{2-51}$$

If we integrate F_λ over all wavelengths we get the bolometric flux

$$F_{bol} = \int_0^\infty F_\lambda(T)\,d\lambda = \sigma T^4 \tag{2-52}$$

Here $\sigma = 5.67 \times 10^{-5}$ erg/cm^2/sec/K^4 is the Stefan-Boltzmann constant. We recognize that $F_{bol} = \sigma T^4$ is the Stefan-Boltzmann Law, the total energy emitted per second from every square cm of a body over the entire EM spectrum. Geometrically, this is the area under any of the curves shown in the diagram above. Also, recall that $L_* = 4\pi R_*^2 \sigma T^4$.

Now consider the case where λ is large. Then $\exp(hc/\lambda kT)$ and $e^x \cong 1 + x + \ldots..$, where $x = hc/\lambda kT$, so

$$B_\lambda \cong (2hc^2/\lambda^5)\ [\ 1/(1+x-1)] \cong (2hc^2/\lambda^5)[\ 1/(hc/\lambda kT)] \cong 2ckT/\lambda^4,$$

which is the Rayleigh-Jeans approximation. Now consider λ to be small and T not very large, then $hc/\lambda kT >> 1$. So

$$B_\lambda \cong (2hc^2/\lambda^5)\ [\ 1/\exp(hc/\lambda kT)] \cong (2hc^2/\lambda^5)\ e^{-hc/\lambda kT},$$

which is the Wien approximation. See Fig. 2, in Chapter 2A.

Do RJP-34, 35, 36.