RJP-708

Evaluate $\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx = \langle \sin mx \cos nx \rangle_{0 \to 2\pi}$

From trigonometric tables:

$$\sin mx \cos nx = \tfrac{1}{2}\left[\sin(mx+nx) + \sin(mx-nx)\right]$$

$$= \tfrac{1}{2}\left[\sin(m+n)x + \sin(m-n)x\right]$$

let $k = m+n$ and $l = m-n$, then

$$\sin mx \cos nx = \tfrac{1}{2}\left[\sin kx + \sin lx\right]$$

So
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \tfrac{1}{2}\left[\sin(kx) + \sin(lx)\right]dx$$

$$= \frac{1}{4\pi}\left\{\int_{-\pi}^{\pi}\sin(kx)\,dx + \int_{-\pi}^{\pi}\sin(lx)\,dx\right\}$$

$$= \frac{1}{4\pi}\left\{\left(\frac{-1}{k}\right)\cos kx\Big|_{-\pi}^{\pi} + \left(-\frac{1}{l}\right)\cos lx\Big|_{-\pi}^{\pi}\right\}$$

Now k and l are always integers, so

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx = \frac{1}{4\pi}\left\{\left[-\frac{1}{k}(\cos k\pi - \cos -k\pi)\right] - \left[\frac{1}{l}(\cos l\pi - \cos -l\pi)\right]\right\}$$

$$= \frac{1}{4\pi}\left\{\left[-\frac{1}{k}\right]\underbrace{\left[-1 - (-1)\right]}_{=0} - \left[\frac{1}{l}\right]\underbrace{\left[(-1 - (-1))\right]}_{=0}\right\}$$

$$= 0 \qquad k \neq n$$