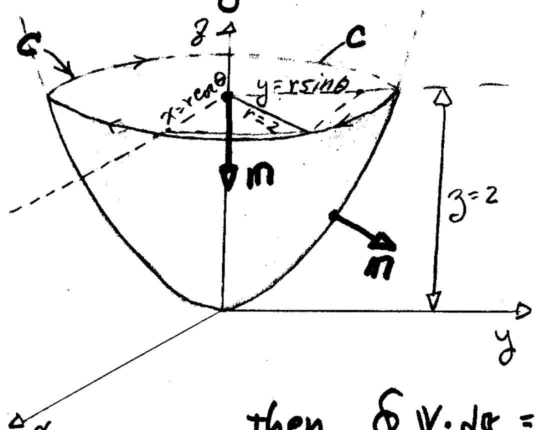
RJP-693

Verify Stokes's Theorem for $\mathbb{V} = 3y\,\mathbf{i} - xz\,\mathbf{j} + yz^2\,\mathbf{k}$ and the surface of the paraboloid $2z = x^2 + y^2$ bounded by $z = 2$.

The bounding curve is the circle found for $z=2$, C. i.e. $x^2+y^2=4$. Use the parametric equations



$x = r\cos\theta$
$y = r\sin\theta$
$r = 2$

around curve C, so

$x = 2\cos\theta$
$y = 2\sin\theta$
$z = 2$

Also $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$

then $\oint_C \mathbb{V}\cdot d\mathbf{r} = \oint [3y\,dx - xz\,dy + yz^2\,dz]$

but along curve C, $dz = 0$. Transforming:

$$\oint_C \mathbb{V}\cdot d\mathbf{r} = \int_{2\pi}^{0} [3(2\sin\theta)\underbrace{(-2\sin\theta)d\theta}_{dx} - (2\cos\theta)(2)\underbrace{(2\cos\theta)d\theta}_{dy}]$$

where the direction of integration is clockwise along C, so that the right-hand rule gives agreement with $\mathbf{n}$ into the paraboloid, i.e $-\mathbf{k} = \mathbf{n}$. Then

$$\oint_C \mathbb{V}\cdot d\mathbf{r} = \int_0^{2\pi} (12\sin^2\theta + 8\cos^2\theta)\,d\theta = 20\pi, \quad \text{since } \int_0^{2\pi} \begin{matrix}\sin^2\theta\\ \cos^2\theta\end{matrix}\, d\theta = \pi$$

Now do $\int \nabla\times\mathbb{V}\cdot\mathbf{n}\,d\sigma$ over circle $x^2+y^2=4$ in $z=2$ plane.

$\nabla\times\mathbb{V} = (z^2+x)\,\mathbf{i} - (z+3)\,\mathbf{k} = (4+x)\,\mathbf{i} - 5\mathbf{k}$ since $z=2$

$(\nabla\times\mathbb{V})\cdot(-\mathbf{k}) = 5$

$\int (\nabla\times\mathbb{V})\cdot\mathbf{n}\,d\sigma = 5\iint d\sigma = 5\pi r^2 = 5\pi(2)^2 = 20\pi$

Stokes is correct again, by golly! Q.E.D.