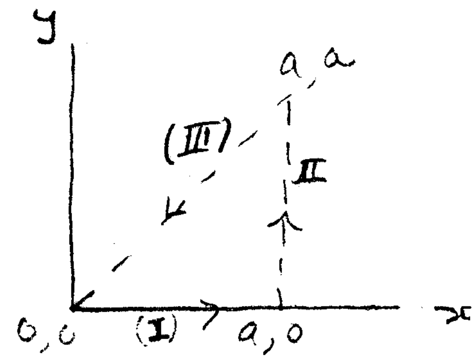
RJP 690 pg. 1

Find $\oint \vec{B} \cdot d\vec{l}$, if $\vec{B} = 2xy\,\hat{i} + x^2\hat{j}$ around the path shown. Verify your result by using Stokes' Theorem, ie, evaluate

$$\int (\vec{\nabla} \times \vec{B}) \cdot \vec{n}\, d\sigma$$



$$\overbrace{\oint \mathbf{B} \cdot d\vec{l}}^{L.H.S} = \overbrace{\int (\nabla \times \mathbf{B}) \cdot \mathbf{n}\, d\sigma}^{R.H.S.}$$

L.H.S.: $\int \vec{B} \cdot d\vec{l} = \int [(2xy\,\hat{i} + x^2\hat{j}) \cdot \hat{i}\,dx + \hat{j}\,dy]$
$= \int (2xy\,dx + x^2 dy)$

(I) $x \to 0$ to a, $y = 0$, $dy = 0$

Path (I) $\int \vec{B} \cdot d\vec{l} = \int_{x=0}^{a} (2xy\,dx + x^2 dy)$

$= \int_{x=0}^{a} (0 + 0) = 0$

(II) $y \to 0$ to a, $x = a$, $dx = 0$

(II) $\int \mathbf{B} \cdot d\vec{l} = \int_{y=0}^{a} [\; a^2 dy] = [y a^2]_0^a = a^3$

(III) $x \to a$ to 0, $y = x$, $dy = dx$

(III) $\int \mathbf{B} \cdot d\vec{l} = \int_{x=a}^{0} [\,2x^2 dx + x^2 dx\,]$

$\int \mathbf{B} \cdot d\bar{l} = \int_{x=a}^{0} [\,3x^2 dx\,]$

$\int \mathbf{B} \cdot d\bar{l} = [\; x^3 \;]_a^0$

Adding results for each path, $\int \mathbf{B} \cdot d\mathbf{l} = -a^3$

$(\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) = 0 + a^3 - a^3 = 0$

For Right Hand Side of Stokes's Theorem:

$$R.H.S = \int (\vec{\nabla} \times \vec{B}) \cdot \vec{n}\, d\sigma$$

$$\vec{\nabla} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & x^2 & 0 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial z}(x^2) - 0\right) - \hat{j}\left(\frac{\partial}{\partial z}(2xy) - 0\right) + \hat{k}\left(\frac{\partial}{\partial y}(2xy) - \frac{\partial}{\partial x}(x^2)\right)$$

$$= \hat{i}(0) - \hat{j}(0) + \hat{k}(2x - 2x) = 0$$

$\therefore$ $R.H.S = \int (\vec{\nabla} \times \vec{B}) \cdot \hat{n}\, d\sigma = 0$, which is the same result found for L.H.S.