RJP-521

Integral over a volume, e.g. $\int \rho \, dV = \text{mass}$, or $I_x = \int \ell^2 \rho \, dV$

Find the moment of inertia of the solid of revolution of $y = x^2$ around x-axis when $\rho = \text{const}$.

$$I_x = \int dI_x = \int \ell^2 dM = \int \ell^2 \rho \, dV$$

ℓ = distance from dM to x-axis. In general $\ell = \sqrt{y^2 + z^2}$

So $\quad I_x = \rho \int \ell^2 dV = \rho \iiint (y^2 + z^2) \, dx \, dy \, dz$

The limits are the same as those given in equation (3.10)

$$I_x = \rho \int_0^1 dx \int_{-x^2}^{x^2} dz \int_{y=-\sqrt{\ }}^{+\sqrt{\ }} (z^2 + y^2) \, dy$$ where $\sqrt{x^4 - z^2}$ are the limits on y.

Break into 2 separate integrals:

$$I_x = \underbrace{\rho \int_{x=0}^{1} dx \int_z dz \int_{-y}^{+y} y^2 \, dy}_{I_{x_1}} + \underbrace{\int_{x=0}^{1} dx \int_z z^2 dz \int_y dy}_{I_{x_2}}$$

$$I_{x_1} = \rho \int_x dx \int_z \left. \frac{y^3}{3} \right|_{-y=\sqrt{\ }}^{y=\sqrt{\ }} dz = \rho \int_x dx \int_z \left[\frac{(x^4 - z^2)^{3/2}}{3} + \frac{(x^4 - z^2)^{3/2}}{3} \right] dz$$

$$I_{x_1} = \frac{2\rho}{3} \int_x dx \int_z (x^4 - z^2)^{3/2} dz$$

$$= \frac{2\rho}{3} \int dx \left\{ \frac{1}{4} \left[z\sqrt{(x^4 - z^2)^3} + \frac{3x^4 z}{3} \sqrt{x^4 - z^2} + \frac{3x^8}{2} \sin^{-1}\left(\frac{z}{x^2}\right) \right]_{-x^2}^{x^2} \right\}$$

$$I_{x_1} = \frac{\rho}{6} \int_0^1 \left[\frac{3x^8}{2} \underbrace{\sin^{-1}(1)}_{=\pi/2} - \frac{3x^8}{2} \underbrace{\sin^{-1}(-1)}_{=-\pi/2} \right] dx = \frac{3\rho}{6} \int_0^1 \left(\frac{\pi}{2}\right) x^8 dx$$

$$I_{x_1} = \frac{\pi\rho}{4} \int_0^1 x^8 dx = \frac{\pi\rho}{4} \left[\frac{x^9}{9} \right]_0^1 = \frac{\pi\rho}{36}$$

Now $I_{x_2} = \rho \int_0^1 \int_z z^2 \left[y \right]_{-\sqrt{\;}}^{+\sqrt{\;}} dz\, dx = 2\rho \int_0^1 \int_z \left[z^2 \sqrt{x^4 - z^2} \right] dz\, dx$

$$I_{x_2} = 2\rho \int_0^1 \left\{ -\frac{3}{4}(x^4 - z^2)^{3/2} + \frac{x^4}{8}\left[z\sqrt{x^4 - z^2} + x^4 \sin^{-1}\left(\frac{z}{x^2}\right) \right] \right\}_{z=-x^2}^{z=x^2} dx$$

$$I_{x_2} = 2\rho \int_0^1 \left[\frac{x^8}{8} \underbrace{\sin^{-1}\left(\frac{x^2}{x^2}\right)}_{=\pi/2} - \frac{x^8}{8} \underbrace{\sin^{-1}\left(-\frac{x^2}{x^2}\right)}_{=-\pi/2} \right] dx$$

$$I_{x_2} = \frac{\rho}{4} \int_0^1 \left[x^8 \left(\frac{\pi}{2} + \frac{\pi}{2} \right) \right] dx = \frac{2\pi\rho}{8} \int_0^1 x^8\, dx = \frac{\pi\rho}{4} \left[\frac{x^9}{9} \right]_0^1 = \frac{\pi\rho}{36}$$

$$I_x = I_{x_1} + I_{x_2} = \frac{\pi\rho}{36} + \frac{\pi\rho}{36} = \frac{\pi\rho}{18}$$

Now $M = \rho V = \rho\left(\frac{\pi}{5}\right)$. So $\rho = 5M/\pi$

Now we may write I_x in terms of M as

$$I_x = \left(\frac{5M}{\pi}\right)\left(\frac{\pi}{18}\right) = \frac{5}{18} M$$