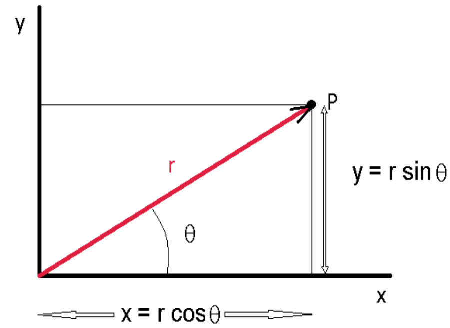
CHAPTER 6

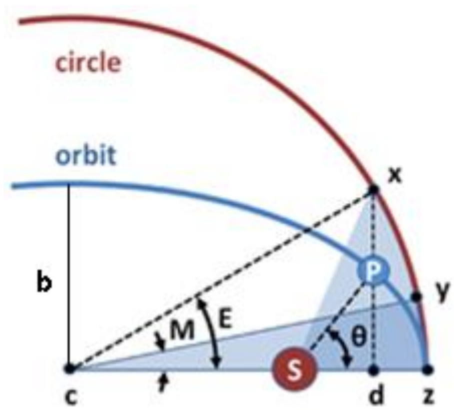
Celestial Mechanics - Computing Planetary Orbits and Positions

6-A. Orbital Position as a function of time



With the help of his three laws, Kepler devised a method for computing the position of a planet in its orbit as a function of time, using polar coordinates r and θ (some authors use υ, upsilon, instead of theta). The diagram to the right illustrates how the rectangular coordinates x and y are related to the polar coordinates r and θ.

Computing θ or υ for a given time t is the most challenging part of the problem. Kepler's method involves the solution of a transcendental equation that he derived now called Kepler's equation. The next diagram to the right shows the geometric construction for Kepler's calculation of θ or υ. The Sun (located at the focus of the elliptical orbit) is labeled *S* and the planet is *P*. The auxiliary, red circle is an aid to the calculation. The point c is on the line of apsides and is the common center of the auxiliary circle and the elliptical orbit. The line *xd* is perpendicular to the semi-major axis and through the planet *P*. The blue shaded sectors are arranged to have equal areas by positioning of point *y*. The Keplerian method assumes an elliptical orbit and the four points:



S, the Sun (at one focus of ellipse);
z, the perihelion point
c, the common center of the ellipse and auxiliary circle
P, the planet

Also,

a = cz is distance between the center of the ellipse, c, and the perihelion point, z, and is equal to the s**emi-major axis.** The latter is also the radius of the auxiliary circle.

ε = cS/a is the **eccentricity**,

$b = a\sqrt{1-\varepsilon^2}$, is the **semi-minor axis**,

r = SP is the distance between Sun and planet. This is the radius vector.

$\theta =$ the angle zSP and is called the **true anomaly**, that is, the direction to the planet as seen from the Sun relative to the perihelion position.

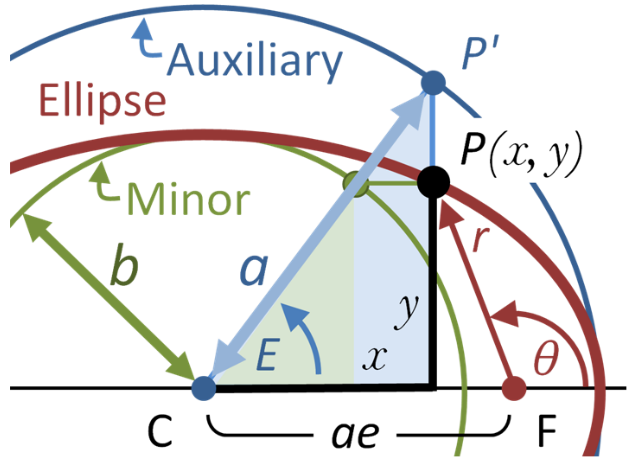
M is the angle zcy and is called the **mean anomaly**. It is the value of the angle in polar coordinates relative to the center of what is called the auxiliary circular orbit. The radius of the latter is equal to the semi-major axis of the elliptical orbit.

The procedure for calculating the heliocentric polar coordinates *r* and *θ* (or υ) of a planetary position as a function of the time *t* since perihelion, and the orbital period *P*, is done in four steps:

Step 1. Compute the **mean anomaly *M*** (in radians) from the formula:

$$M = \frac{2\pi t}{P}$$

Here P is the period of revolution and t is the time since perihelion passage, when M = θ = 0. P and t must be in the same units.



Step 2. Compute the **eccentric anomaly, *E***, by solving Kepler's equation:

$$M = E - \varepsilon \cdot \sin E$$

Again, ε is the eccentricity of the ellipse. Notice that M is less than E before aphelion. The greater the value of ε, the greater the difference between E and M. This transcendental equation may be solved by iteration in the form: E = M + ε sin E. E must be in radians. Take the initial value of E to be equal to M. The solution converges after about six iterations.

Step 3. Compute the **true anomaly,** *υ* (*θ in the diagram*), by the equation:

$$\cos \upsilon = \frac{\cos E - \epsilon}{1 - \epsilon \cdot \cos E}$$

or equivalently:

$$\tan \frac{\nu}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2}.$$

Here e is the eccentricity. Therefore:

$$\upsilon = 2 \arctan \left(\sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2} \right)$$

Here one must be wary of the proper quadrant of υ after using the arctangent function. The proper quadrant is better determined by using the equation with the cosine function.

Step 4. Compute the **heliocentric distance** *r* from the first law:

$$r = \frac{p}{1 + \epsilon \cos \upsilon}$$

Here, $p = a(1 - \varepsilon^2)$.

In the special case of a circular orbit, ε = 0, which gives simply *υ* = *E* = *M*.

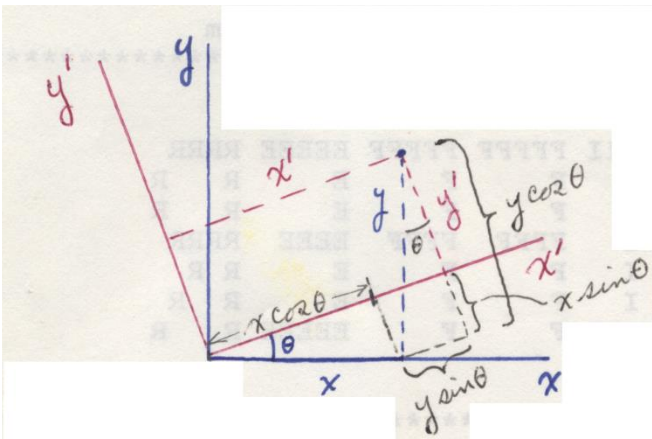
Assignment 6-A 1:

Write a program that computes the position of Mars in its orbit every 10 days starting with the perihelion passage. Plot the results with Excel. Obtain all necessary parameters from: nssdc.gsfc.nasa.gov/planetary/factsheet/**mars**fact.html. Normalize r to the perihelion distance, that is, take r to be 1.00 at perihelion. Derive by hand, how you did this on a separate page and then incorporate the resulting equation into your program.

How would you go about plotting the position of Mars on a rectangular star chart relative to the equatorial coordinate system? In order to answer such a question, we need to discuss how to transform position in one coordinate system to a position in another coordinate system. For example, we want to learn how to transform the position of a planet in a coordinate system in the plane of its orbit to the ecliptic coordinate system and then to the equatorial system.

6-B. Rotation of Coordinates

In many instances in astronomy, we need to transform a position, x, y, and z, in one coordinate system to a position x', y', z' in another coordinate system. The second coordinate system, or prime system, has its axes rotated relative to the original coordinate system by an angle θ. This angle is not to be confused with the true anomaly defined in the previous section. The angle θ now is just some arbitrary angle. First consider the simple case of a two-dimensional, or planar, coordinate system, such as the plane of an orbit. See the diagram below. Using some trigonometry we derive the transformation as shown:



$$x' = x\cos\theta + y\sin\theta$$

$$y' = -x\sin\theta + y\cos\theta$$

Define rotational matrix as:

$$RM = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

A matrix is defined to be an array of numbers that obeys certain laws of operation. We also define the two matrices

$$\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x' \\ y' \end{pmatrix}$$

The matrix operation shown below produces the above transformation equations for x, y to x', y'.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

This operation is called row–column matrix multiplication. That is, each element in a row in the first matrix on the right side of the equal sign is multiplied by each element in the column of the second matrix on the right and the results are added. The result is the set of transformations equations given above. We shall need this kind of operation for transforming the positions of planets in their orbits to any other coordinate system.

In the case of 3 dimensions, the above rotation may be visualized as a rotation around the z-axis, which is perpendicular to the xy plane. In this case we have

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Multiplication of the two matrices on the right yields:

$$x' = x\cos\theta + y\ \sin\theta + 0\cdot z$$
$$y' = -x\sin\theta + y\ \cos\theta + 0\cdot z$$
$$z' = x\cdot 0 + y\cdot 0 + 1\cdot z = z$$

Rotational matrices may be written for rotation around any axis.

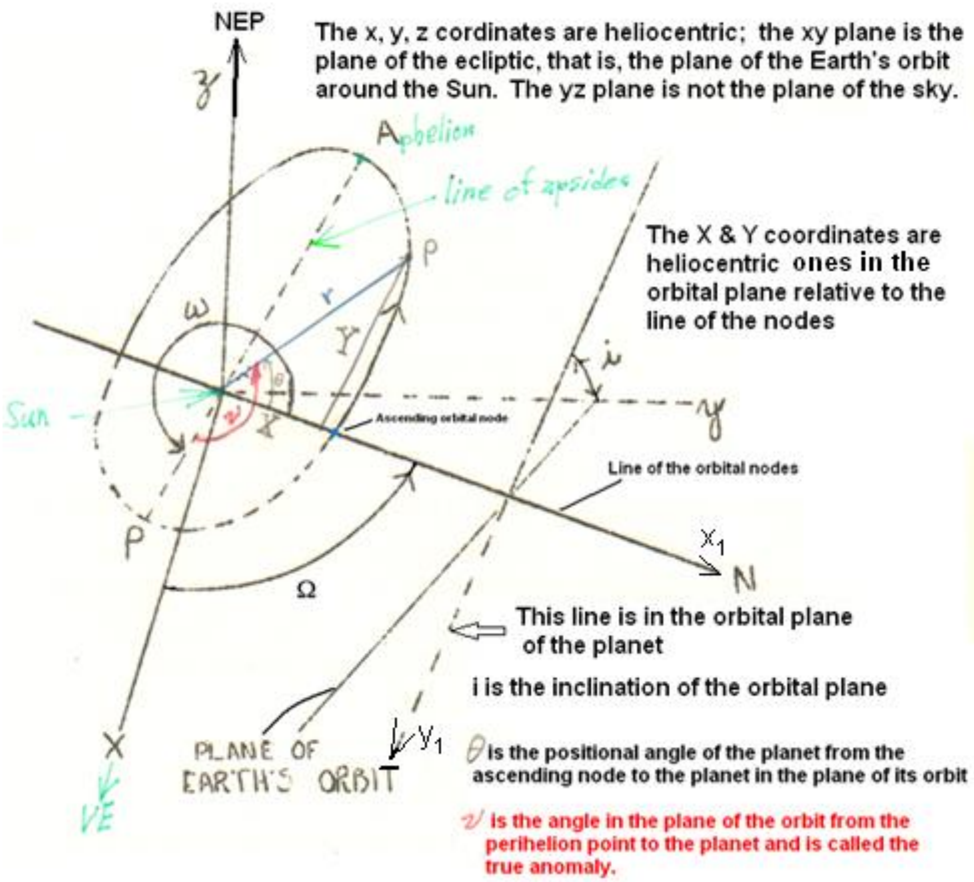
6-C. Orbital Elements

We now turn to defining the orbital elements of a body such as a planet, asteroid, comet, or any satellite moving in orbit about its primary. To do so, one must distinguish between the numbers that:

(1) Describe the position of an orbital plane in space (the orientation elements) relative to some coordinate system,
(2) The elements that describe the size and shape of an orbit,
(3) The elements that describe the orientation of the orbital conic in its plane, and
(4) The elements that describe the position of a body in its orbit at some epoch of time.

Such an orbit is shown relative to the ecliptic plane in the diagram below. In this diagram and the discussion that follows, the angles θ and υ are not the same.

The xy-plane is the plane of the ecliptic, that is, the plane of the Earth's orbit around the Sun



The orbital plane of the planet is inclined to the plane of the ecliptic by the angle i.
The orbital nodes are the points where the orbit intersects the ecliptic plane. The ascending node is the point where the planet crosses the ecliptic plane moving from south to north, relative to the ecliptic poles.
The line connecting the two points of intersection of the orbit with the ecliptic plane is called the line of the node. It is the line SN.
The positive direction of the x-axis is taken as the line from the Sun to the vernal equinox, VE. The z-axis is towards the north ecliptic pole.

For (1), the orientation of the planet's orbit relative to the xyz system is defined by:

i, the inclination of the plane of the orbit to the plane of the ecliptic, and

Ω, the longitude of the ascending node. This angle is in the plane of the ecliptic.

Assignment 6-C 1: Use SKYLAB to find Ω for the planet Mars. You do this by trailing Mars in the SKYLAB program *Skymation*.

For (2), the size and shape of the orbit are determined by the elements:

a, the semi-major axis, and
ε, the eccentricity. For an elliptical orbit, $0 < \varepsilon < 1.0$. For a parabola, $\varepsilon = 1.0$ and for a hyperbola, $\varepsilon > 1.0$.

Finally, for (3), the orientation of conic in its plane is determined by the angle ω, which is the angle measured in the orbital plane from the line of the nodes, SN, to the perihelion point, P. The positive sense of ω is the direction of motion of the planet in its orbit. Be careful so as not to confuse Ω and ω. In most references, ω is called the argument of perihelion. It should not be confused with what is called the longitude of perihelion, ϖ. The latter is defined as $\Omega + \omega$, even though these two angles are not in the same plane. In most references, ϖ and Ω are given, but not ω. However, once ϖ and Ω are known, ω is found by subtraction.

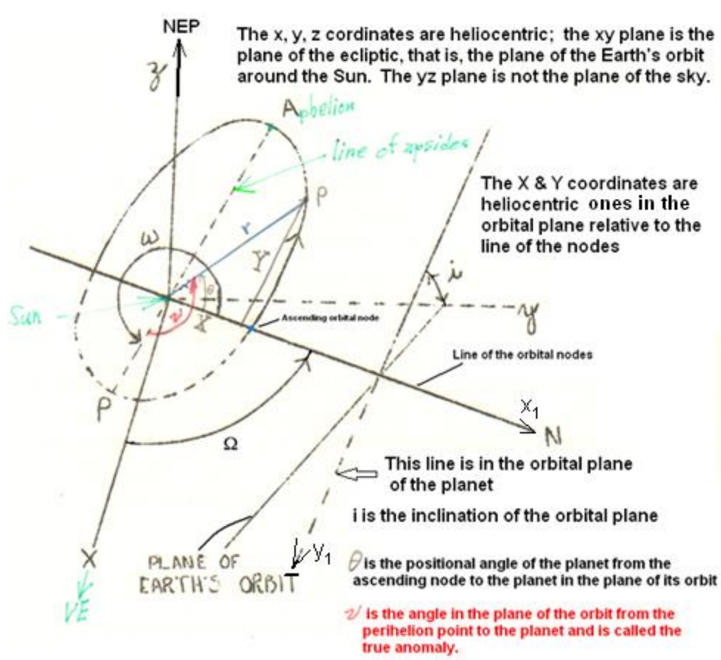
Summary:

True anomaly is υ (in plane of orbit).
Longitude of ascending node is Ω, the angle (in plane of ecliptic) from the VE to the ascending node.
Argument of perihelion is ω, the angle (in plane of orbit) from the line of the nodes to perihelion.
Longitude of perihelion is $\varpi = \Omega + \omega$ (in 2 different planes).
Position angle from ascending node is $\theta = \omega + \upsilon - 2\pi$ (in plane of orbit). This relation is obtained by noting in the diagram above that $2\pi - \omega = \upsilon - \theta$.

6-D. The Geocentric Coordinates of a Planet

In section 6-A, we have shown how to find the position of a planet in its orbit at time t since its perihelion passage, that is, how to find r and υ. The latter are heliocentric polar coordinates in the plane of the orbit. We now turn our attention to computing the geocentric Cartesian coordinates of a planet in the equatorial system. In order to accomplish this we need to make several transformations of coordinates.

We define the orbital plane to be the x_1y_1 plane with the x_1-axis to be along the line of the nodes. Hence, the z_1-axis is perpendicular to the orbital plane. (In the diagram below, X is along the x_1-axis and Y is along the y_1-axis).



Then we have:

$$x_1 = r \cos \theta$$
$$y_1 = r \sin \theta$$
$$z_1 = 0,$$

where $\theta = \upsilon+\omega-2\pi$, as shown in the diagram above. However, $\cos \theta$ and $\sin \theta$ are the same values as the cos and sin of $\upsilon + \omega$, since these functions are periodic in 2π. The above are the heliocentric Cartesian coordinates of a planet in the plane of its orbit, relative to the line of the nodes

Next we transform these coordinates to a coordinate system, $x_2y_2z_2$, where x_2 is also in the direction of the line of the nodes. However, the x_2y_2 plane is now the plane of the ecliptic with the z_2 direction to be towards the north ecliptic pole. This is a coordinate system that is rotated around the x_1-axis by the orbital inclination, - i. Hence,

$$x_2 = x_1 = r \cos \theta$$
$$y_2 = y_1 \cos i = r \sin \theta \cos i$$
$$z_2 = - y_1 \sin i = y_1 \sin i,$$

since cos (-i) = cos (i) and sin (-i) = - sin (i). These are still heliocentric Cartesian coordinates but in the plane of the ecliptic not the plane of the planet's orbit.

Now we perform a third rotation of coordinates. This time we rotate backwards through angle Ω around the z_2-axis until the x_2-axis coincides with the direction to the vernal equinox. The $x_3y_3z_3$ coordinate system is still in the ecliptic system, but the x_3-axis is towards the vernal equinox instead of the point on the ecliptic where the ascending node of the planet's obit is located. Hence, the transformation equations are

$$x_3 = x_2 \cos \Omega - y_2 \sin \Omega$$
$$y_3 = x_2 \sin \Omega + y_2 \cos \Omega$$
$$z_3 = z_{2,}$$

where again, we have used the fact that $\sin(-\Omega) = -\sin \Omega$. Now substitute for the values of x_2, y_2, z_2 from above and factor out r to get:

$$x_3 = r (\cos \theta \cos \Omega - \sin \theta \cos i \sin \Omega)$$
$$y_3 = r (\cos \theta \sin \Omega - \sin \theta \cos i \cos\Omega)$$
$$z_3 = z_2 = r \sin \theta \sin i.$$

It should be recalled that $\theta = \upsilon+\omega$ and the true anomaly are calculated for some time t as explained above in section 6-C. The values of ω and Ω must be looked up, since they are not easy to determine except through observations.

We now transform to the equatorial system, $x_4y_4z_4$, by a rotation around the x_3-axis by the obliquity of the ecliptic, -q. Hence,

$$x_4 = x_3$$
$$y_4 = y_3 \cos q - z_3 \sin q$$
$$z_4 = y_3 \text{ sins } q + z_3 \cos q.$$

These are the heliocentric Cartesian coordinates of a planet in the equatorial system.

Now we must find the position of the planet relative to the Earth, x_o, y_o, z_o, that is, the geocentric coordinates of the planet in the equatorial system. In Fig. 6-D1, the red frame of reference is the geocentric one. We define the heliocentric coordinates of the Earth in the celestial equatorial system to be x_E, y_E, z_E. Then

$$x_o = x_4 - x_E$$
$$y_o = y_4 - y_E$$
$$z_o = z_4 - z_E$$

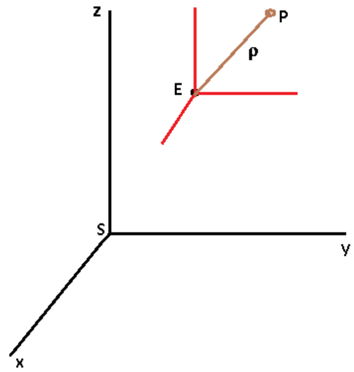


Fig. 6-D1. Heliocentric coordinate system

What are usually given in almanacs are the geocentric equatorial coordinates of the Sun, which are identified as x, y and z. However, we shall refer to them as x_S, y_S, and z_S. (Before 1981, The Astronomical Almanac lists these coordinates as X_o, Y_o, and Z_o.) Now the heliocentric coordinates of the Earth, x_E, y_E, z_E, are just the negative of the geocentric coordinates of the Sun. Hence,

$$x_o = x_4 + x_S$$
$$y_o = y_4 + y_S$$
$$z_o = z_4 + z_S$$

So at this point, we compute x_o, y_o, z_o by looking up x_S, y_S, z_S in ***The Astronomical Almanac***, which is published by the US Naval Observatory. Now from the geometry shown in Fig. 6-D2, we obtain:

$$x_o = x_4 + x_S = \rho \cos \delta \cos \alpha$$
$$y_o = y_4 + y_S = \rho \cos \delta \sin \alpha$$
$$z_o = z_4 + z_S = \rho \sin \delta$$

In Fig. 6-D2, P is the planet and E is the Earth. The x, y, and z axes are the equatorial geocentric coordinate system. The x-axis points to the vernal equinox, the y-axis points to a place on the celestial equator at right ascension 6 hours, and the z-axis points to the north celestial pole.

The above set of equations relate x_o, y_o, z_o to the right ascension and declination of the planet and the distance of the planet from the Earth, ρ.
Hence, we have three equations in three unknowns and we can solve for α, δ, and ρ, once we have computed x_o, y_o, & z_o.

The procedures that we have developed here are essentially the same for computing the position of any body in the solar system, whether it be a planet, comet, or asteroid.

It is also the same procedure for computing the orbit of any body moving around any other body, for example, the orbit of a star in a binary star system.

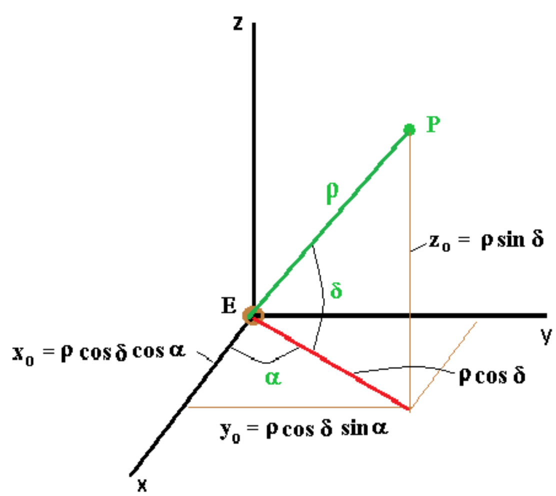


Fig 6-D2. The Geocentric equatorial coordinate system